

Unit - 4

'Graph Theory'

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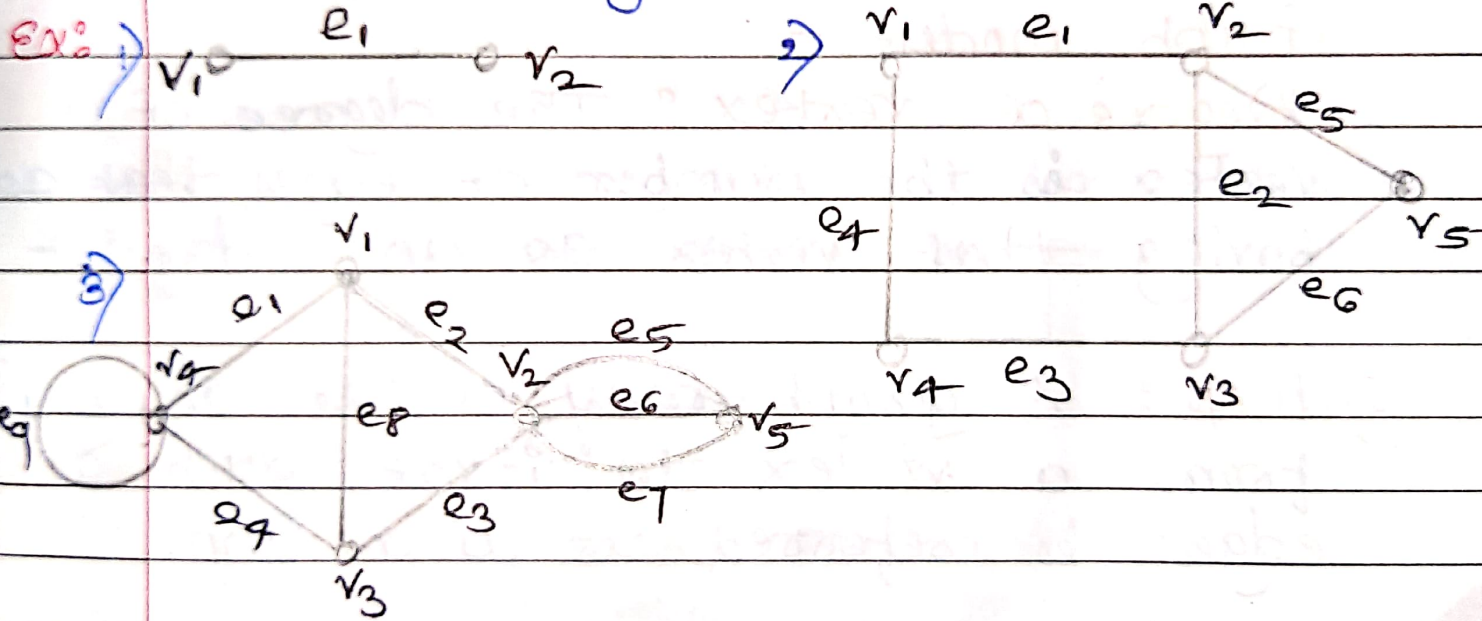
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Graph: A Graph G is mathematical structure consisting of two sets V and E where V is a non-empty set of vertices and E is a non-empty set of edges.

$$G = (V, E)$$

V is the vertex (Node)

E is the Edges.



1) Graph ① : 2 vertices and one edges

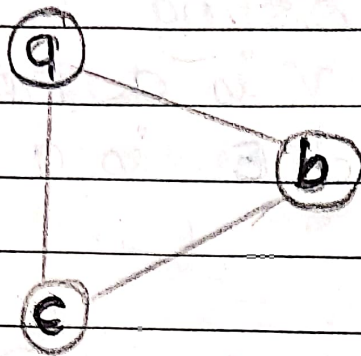
2) Graph ② : 5 vertices and 6 edges

3) Graph ③ : 5 vertices and 9 edges

ex.) Consider a graph $G = (V, E)$

on set $A = \{a, b, c\}$ defined by graph $G = \{(a, b), (b, c), (a, c)\}$

soln

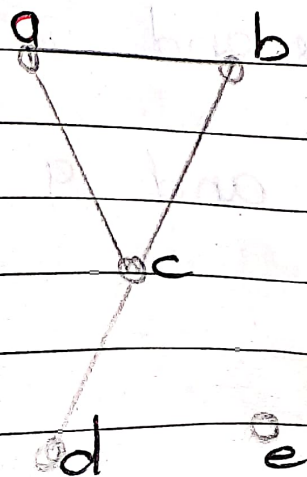


Graph models:-

1) Degree of vertex: The degree of a vertex is the number of edges that are having that vertex as an end point

2) Loop: A graph may contain an edge from a vertex to itself. Such an edge is referred to as a loop.

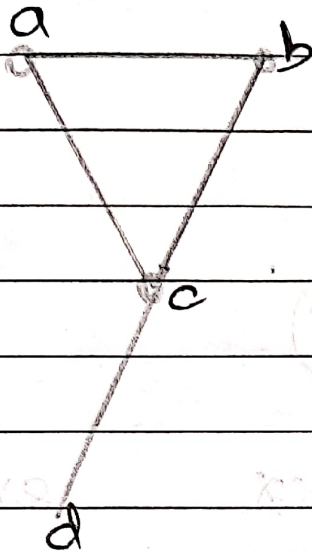
3) Isolated vertex: A vertex with degree 0 is called an isolated vertex.



In the graph e is an isolated vertex.

4) adjacent vertices:

A pair of vertices that determine an edge are adjacent vertices.



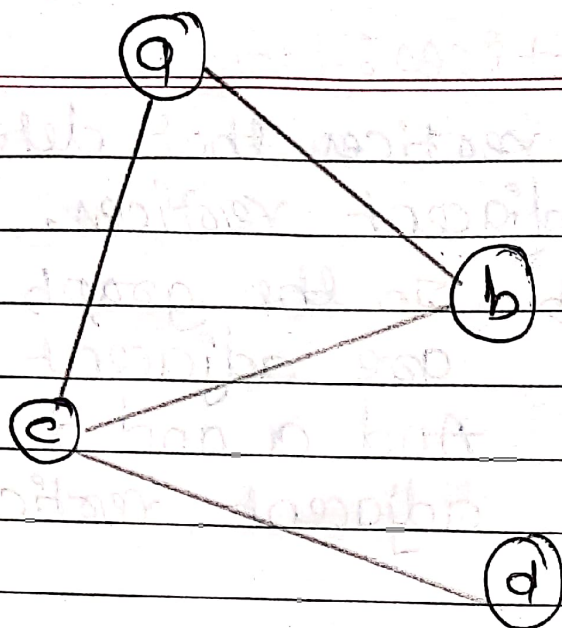
In the graph a and b are adjacent vertices. And a and d are not adjacent vertices.

Degree of Graph: The degree of the graph is the largest vertex degree of the graph.

Ex. 1) Even and Odd vertex:

If the degree of vertex is even then the vertex is called even vertex and if the degree of vertex is odd then the vertex is called odd vertex.

ex. 1) Find degree of vertex even and odd vertex and degree of the graph from the graph given below.



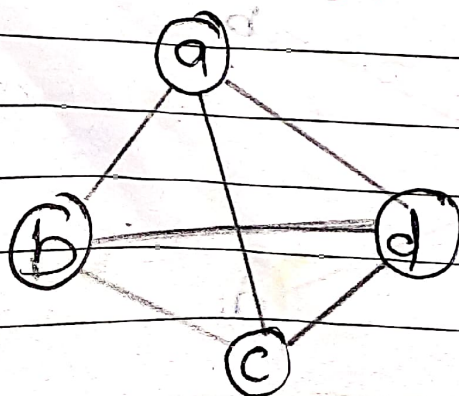
Vertex	degree of vertex	even / odd
a	2	even
b	2	even
c	3	odd
d	1	odd

$\therefore$ degree of graph = 3 (Largest degree in the table)

- 2) Find degree of each vertex even and odd vertex and degree of the graph from the relation given below.

$$G = \{(a, b), (a, c), (a, d), (b, c), (c, d), (b, d)\}$$

So/2

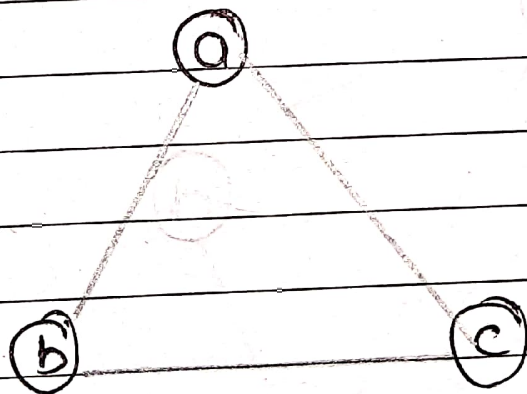


vertex	degree of vertex	even / odd
a	3	odd
b	3	odd
c	3	odd
d	3	odd.

And degree of graph = 3 (largest degree in the table).

Graph Terminology :

- 1) Node (vertex)
- 2) Edge
- 3) Adjacent Node
- 4) Degree of Node (vertex)
- 5) Size of the graph
- 6) path



- 1) Node (vertex) : In that above graph there are 3 vertices. a, b and c
- 2) Edge : In that above graph three

Edges.

- ③ Adjacent Node: In that above graph a and b this is adjacent Node a and c also also the adjacent Node

- ④ Degree of node: No. of edges Connected to that node.

In that above graph degree of node is

$D(A) \rightarrow \text{Degree of } A = 2$

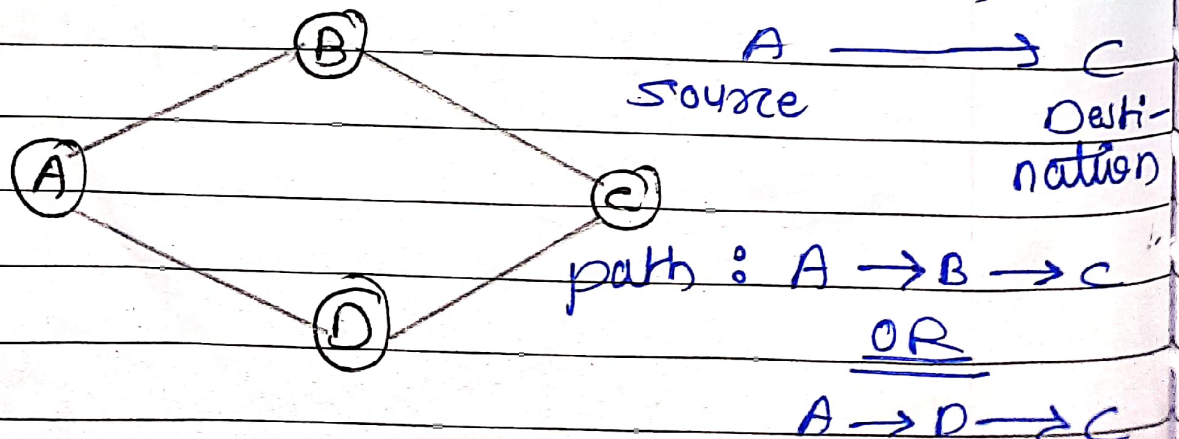
$D(B) \rightarrow \text{Degree of } B = 2$

$D(c) \rightarrow \text{Degree of } c = 2$

- ⑤ Size of the graph: Total Number of edges in graph.

In that above graph
Size = 3

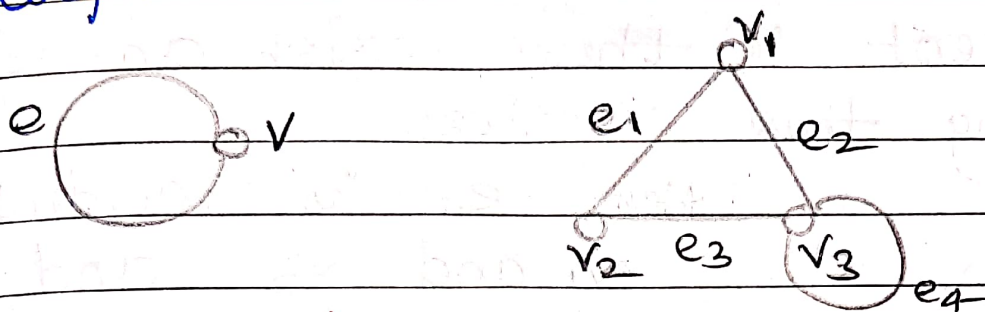
- ⑥ Path: Sequence of vertices from Source vertex to Destination node



1) Self loop in a graph:

If edge having the same vertex as both its end vertices is called Self loop.

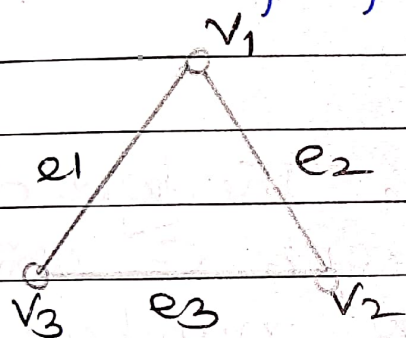
Ex:



2) Proper edge:

An edge which is not self loop is called proper edge.

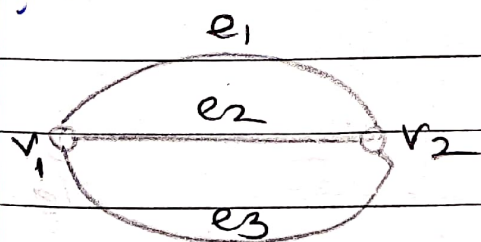
Ex:



e_1 , e_2 and e_3 have no Self loop.

3) Multi edge: A collection of two or more edges having identically end point

Ex:



e_1 , e_2 and e_3 are multi edges.

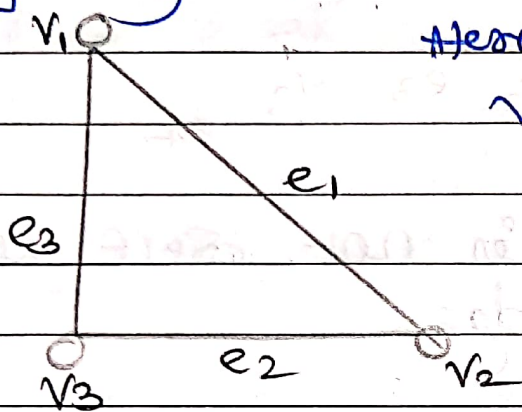
4) Incident or Incidence and Adjacency:

Let e_2 be an edge joining two

vertices v_1 and v_2 then e_1 is said to be incident of v_1 and v_2 .

Two vertices are said to be adjacent if there exist an edge joining this vertices.

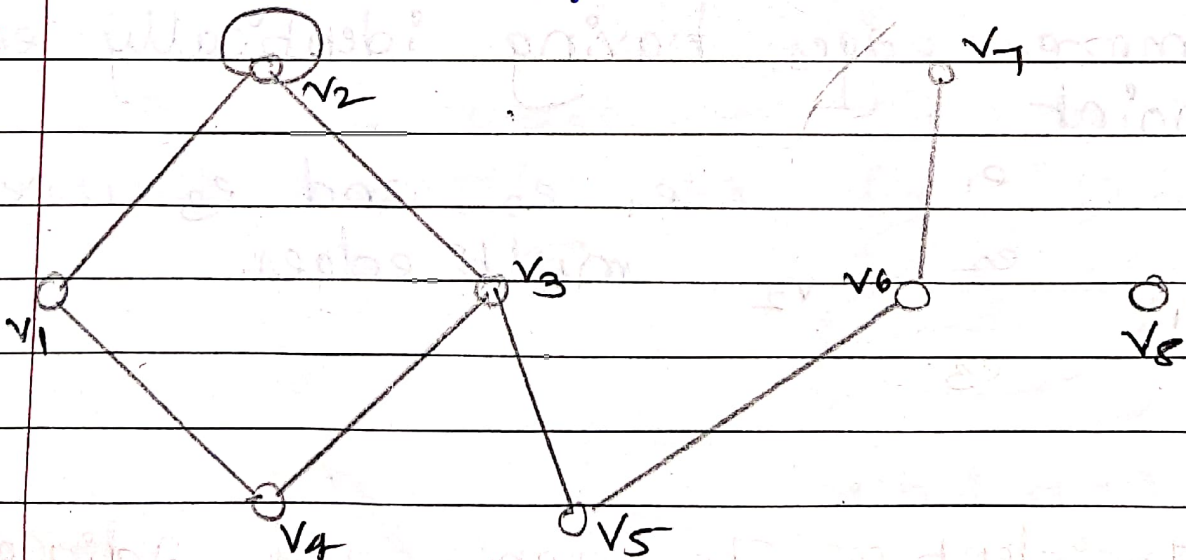
Ex:



Here e_1 is incident to v_1 and v_2 and v_1 and v_2 are adjacent but not v_2 and v_3 .

5) Degree of vertex: The degree of vertex v in a graph G written as $d(v)$ is equal to number of edges which are incident on v with self loop counted twice.

Ex:



Here $D(v_1) = 2$, $D(v_2) = 4$, $D(v_3) = 3$

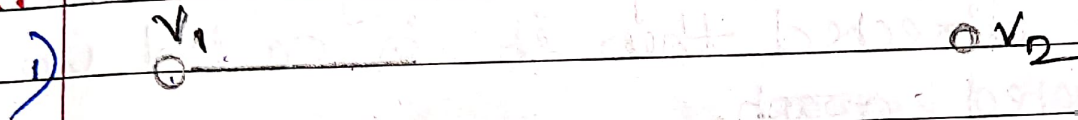
$$D(v_4) = 2, D(v_5) = 2, D(v_6) = 2$$

$$D(v_7) = 1, D(v_8) = 0$$

6) Isolated vertex and pendant vertex:
A vertex having 0 is called isolated vertex.

A vertex having 1 is called pendant vertex:

ex:



Special Types of Graphs:

1) Trivial Graph: A graph consisting only one vertex and no edge.

ex:



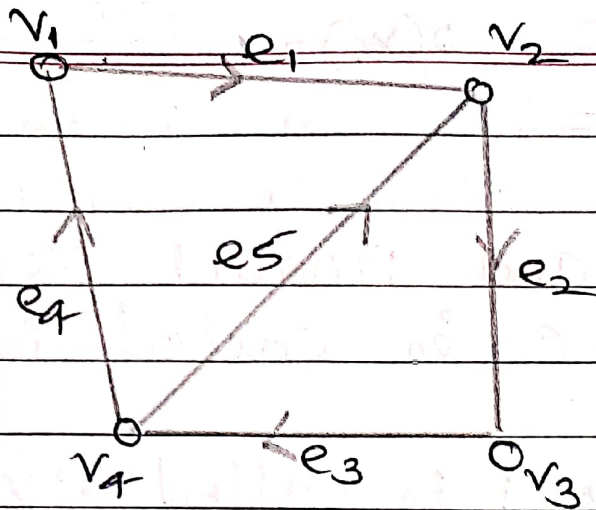
2) Null Graph: A graph consisting n vertices and no edge.

ex:

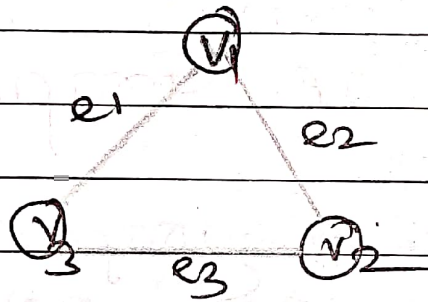


3) Directed Graph: A graph consist the direction of edges. This is called directed graph.

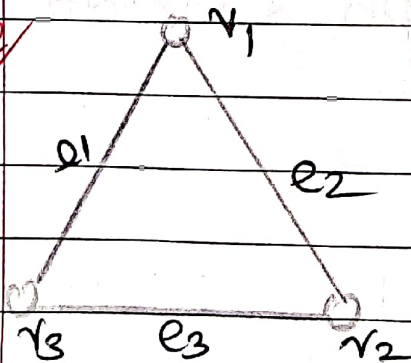
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Undirected Graph: A graph which is not directed then it is called undirected graph.

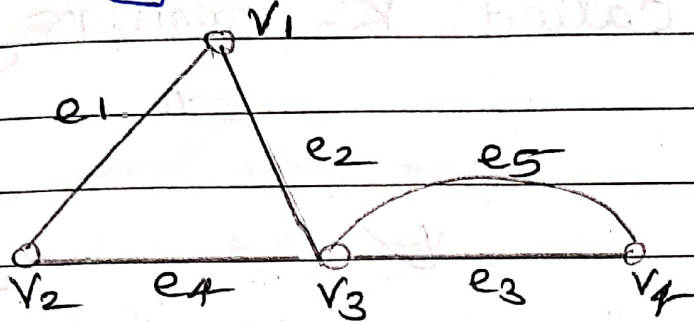


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Simple Graph: A graph does not contain any self loop and multi edge.

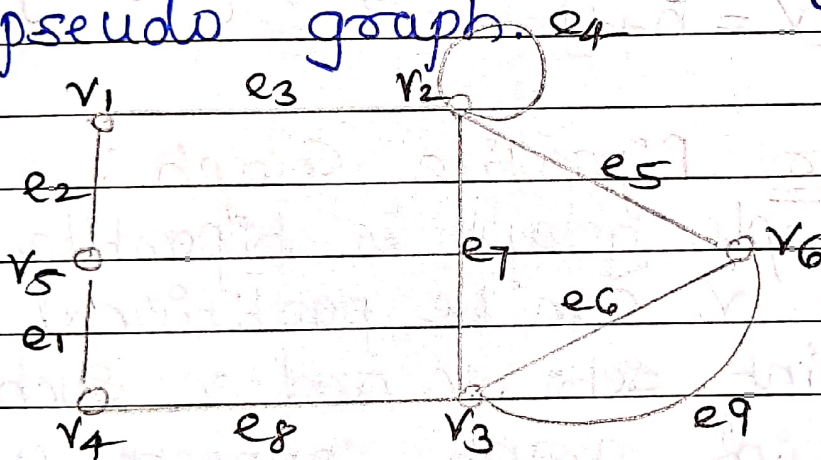


Multi Graph:

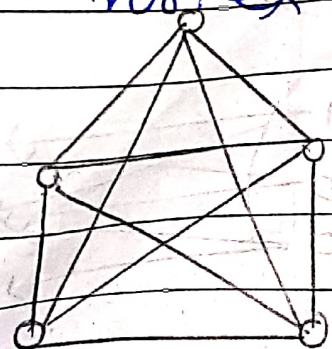
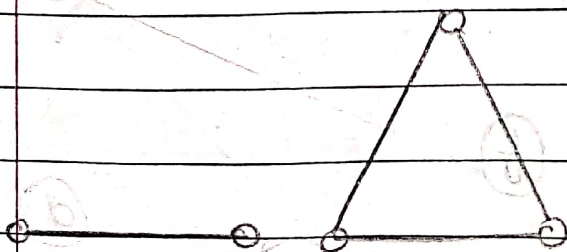
A graph does not contain any self loop but contain multiedge is called multigraph.



7) Pseudo Graph: A graph contain both self loop and multiedge is called pseudo graph.

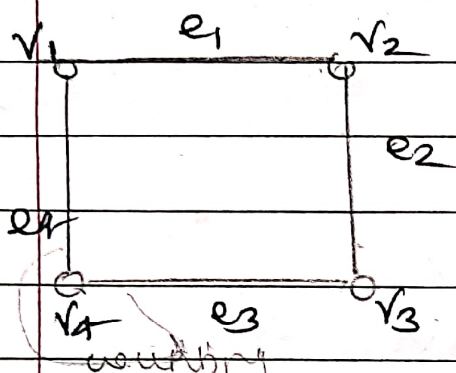


8) Complete Graph: A simple graph is said to be complete if each vertex is connected to every other vertex.

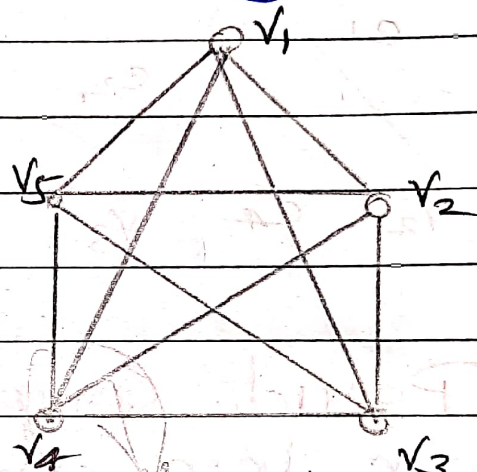


9) Regular Graph:

A graph G is said to be regular if every vertex has the same degree. If degree of each vertex of graph G is k then it is called k -regular graph.



2 Regular graph



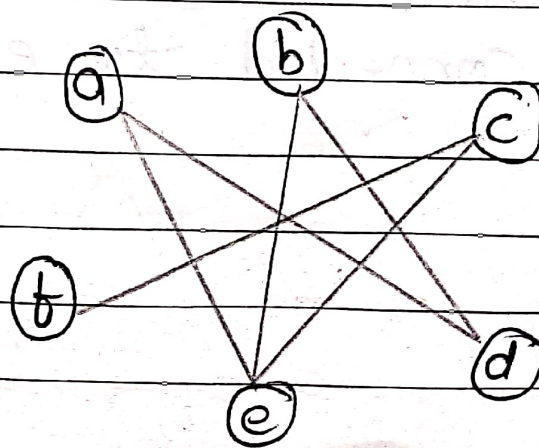
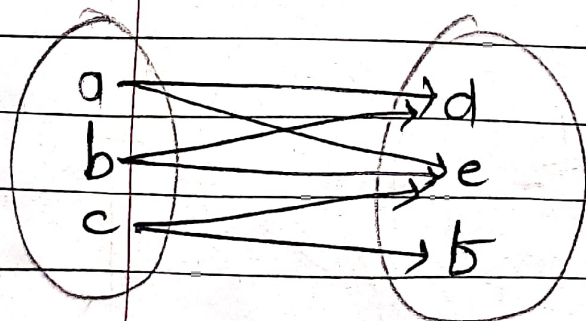
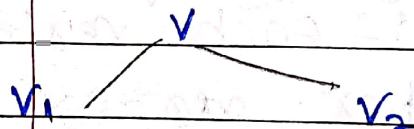
4 regular graph

Note: vertices of n -regular graph are $[V = n+1]$

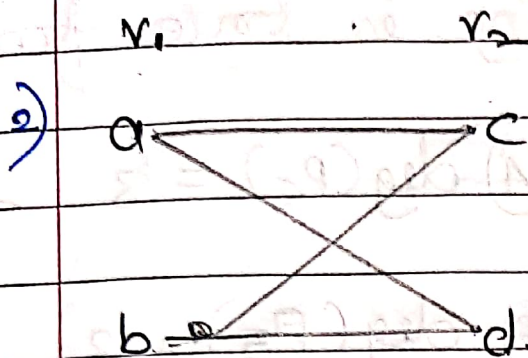
10) Bigraph or Bipartite Graph:

A simple graph is bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that each edge of graph connects a vertex of V_1 to a vertex in V_2 .

ex: 1) $V = \{a, b, c, d, e, f\}$

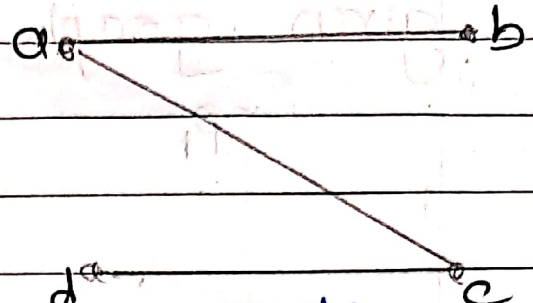


Note: No edge connects either two vertices in V_1 or two vertices in V_2



$$V_1 = \{a, b\}$$

$$V_2 = \{c, d\}$$

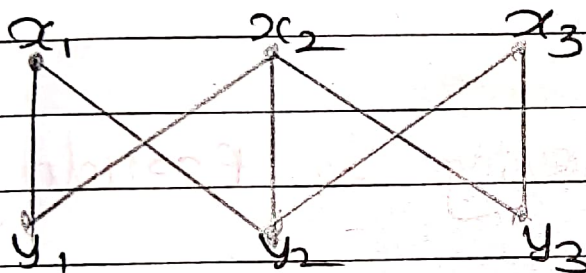


$$V_1 = \{a, d\}$$

$$V_2 = \{b, c\}$$

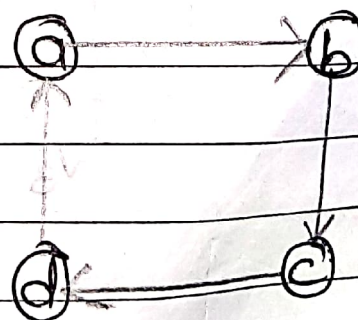
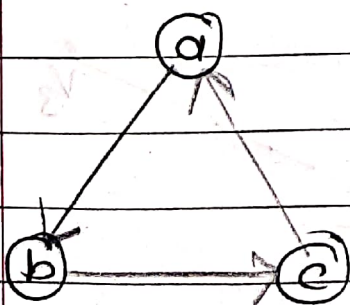
3) $V_1 = \{x_1, x_2, x_3\}$

$$V_2 = \{y_1, y_2, y_3\}$$



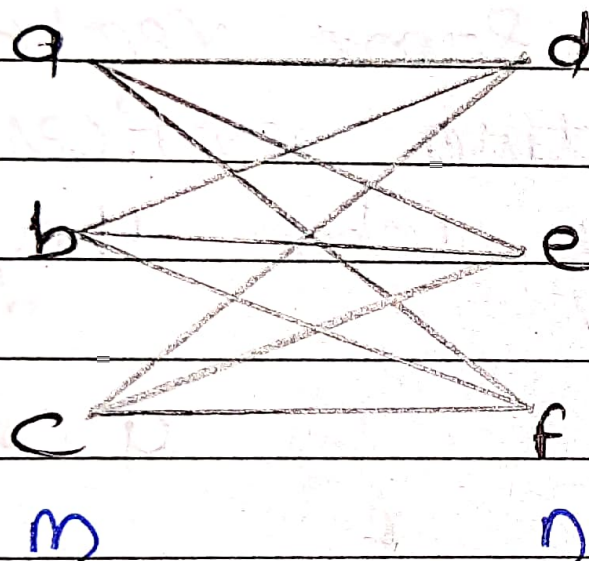
1) Cyclic or Cycle graph: If a directed graph consists of a single cycle is called Cycle graph.

A Cycle graph with n vertices is denoted by C_n

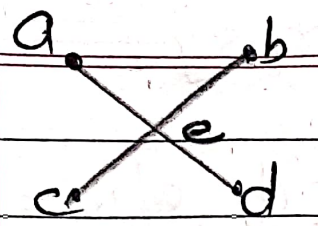


12) Complete Bigraph or Complete Bipartite Graph: A bipartite graph in which every vertex of one set is connected with every vertex of another set. It is denoted by $K_{m,n}$

Ex:)

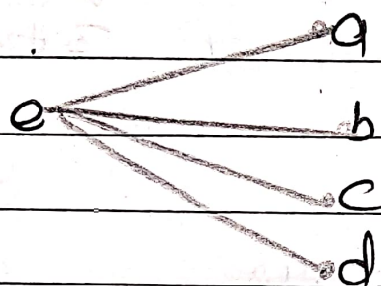


$K_{3,3}$

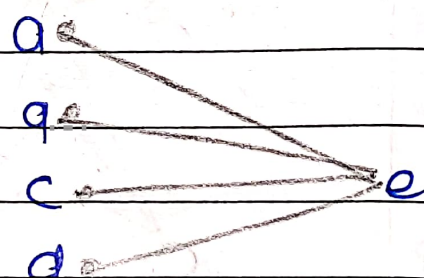
2)  Is the graph is complete Bipartite graph

soln

e is connected to all a, b, c, and d mean



OR



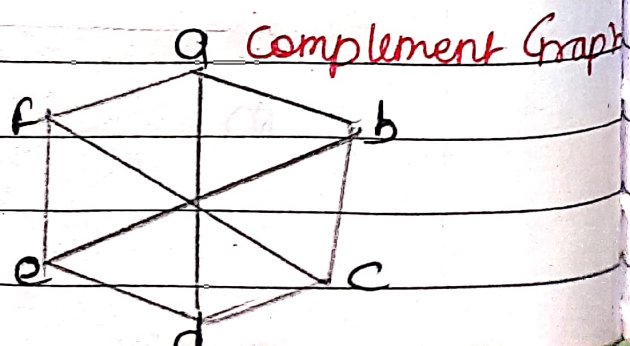
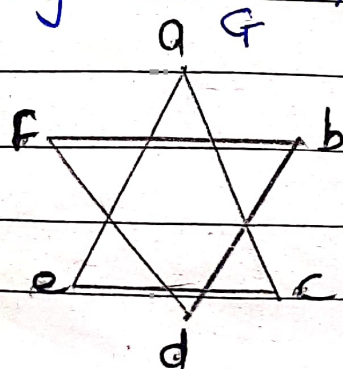
Then

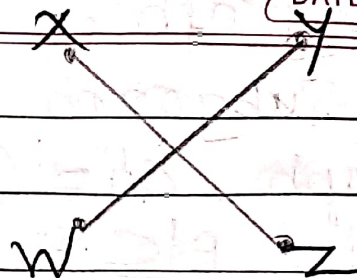
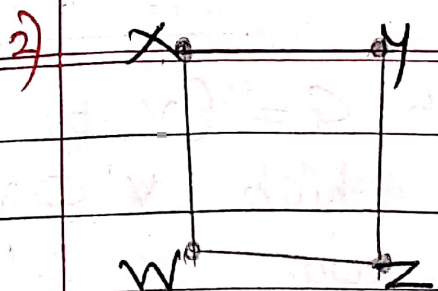
$V_1 = \{e\}$ and $V_2 = \{a, b, c, d\}$

It is a complete Bipartite graph because each vertex of V_1 is connected to each vertex of V_2 .

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3) Complement graph: The Complement of a graph G is defined as a Simple graph with the same vertex set as G and where two vertices u and v are adjacent only when they are not adjacent in G .

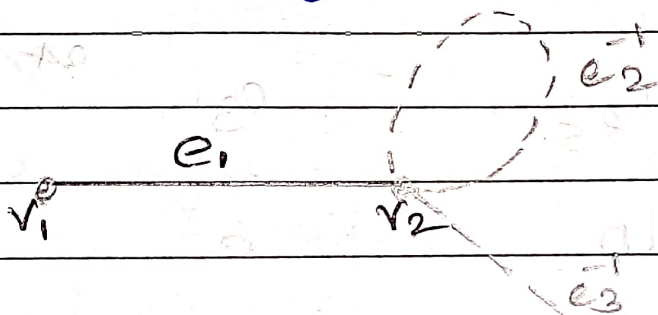
ex: i)





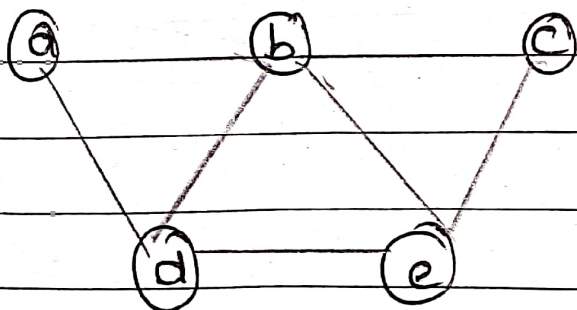
Imp 14) 2019 2M **General graph:** A general graph G is a mathematical structure consisting of two sets V and E where V is a finite non-empty set of vertices and E set of original edges and inverse edges.

Ex)



Imp 15) **Adjacency graph:** In a graph two vertices are said to be adjacent if there is an edge between the two vertices. Here the adjacency of vertices is maintained by the single edge that is connecting those two vertices.

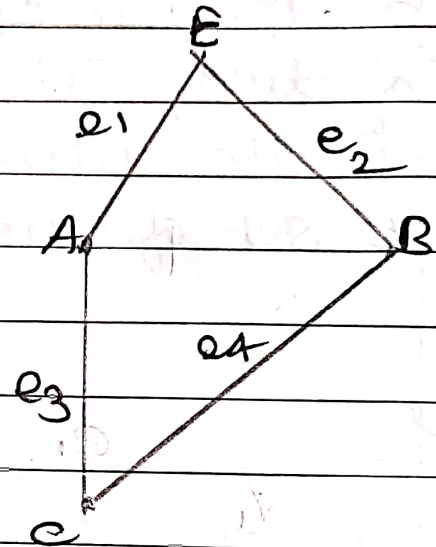
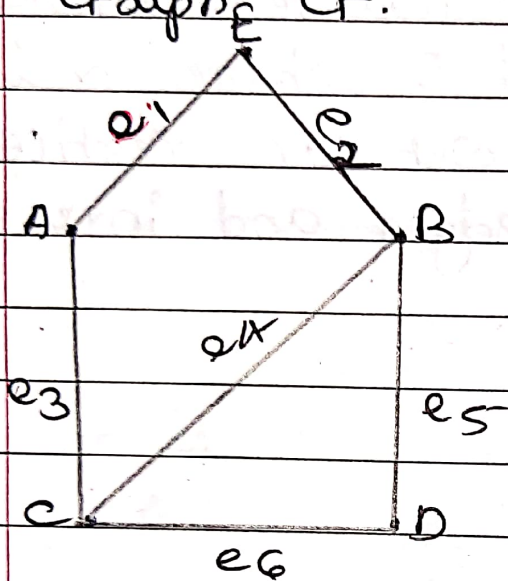
ex.



Imp 2002 2021 16 Subgraph :

A Subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ in which $V' \subseteq V$ and $E' \subseteq E$ and each edge of G' have the same end vertices in G' as in graph G .

Ex : Graph G .



Graph G .

$$V = \{A, B, C, D, E\}$$

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

then $V' \subseteq V$ and

Graph G'

$$V' = \{A, B, C, D, E\}$$

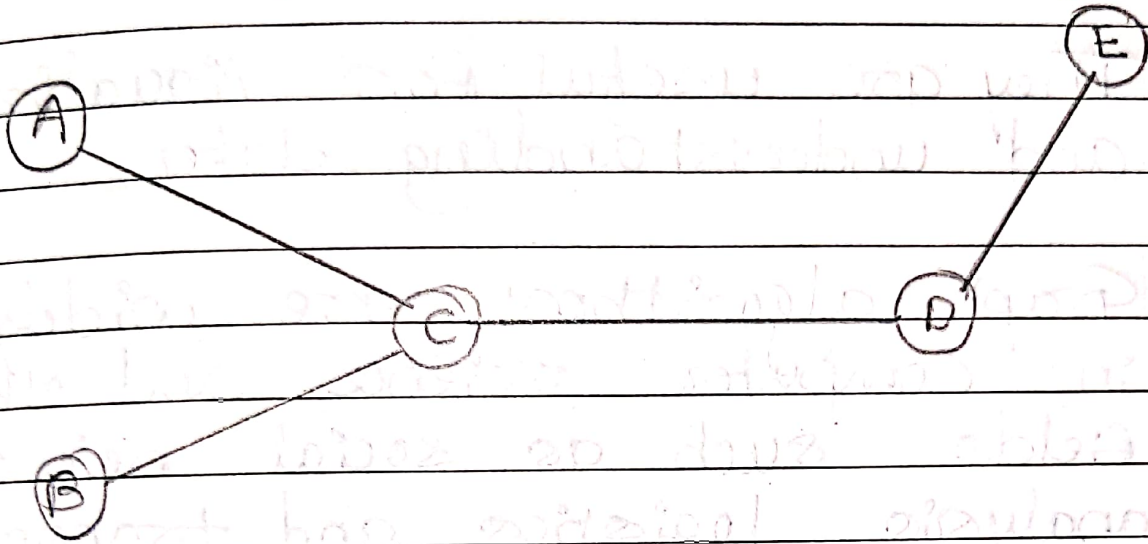
$$E' = \{e_1, e_2, e_3, e_4\}$$

$$E' \subseteq E$$

17) finite graph:

A graph is said to be finite if it has a finite number of vertices and a finite number of edges.

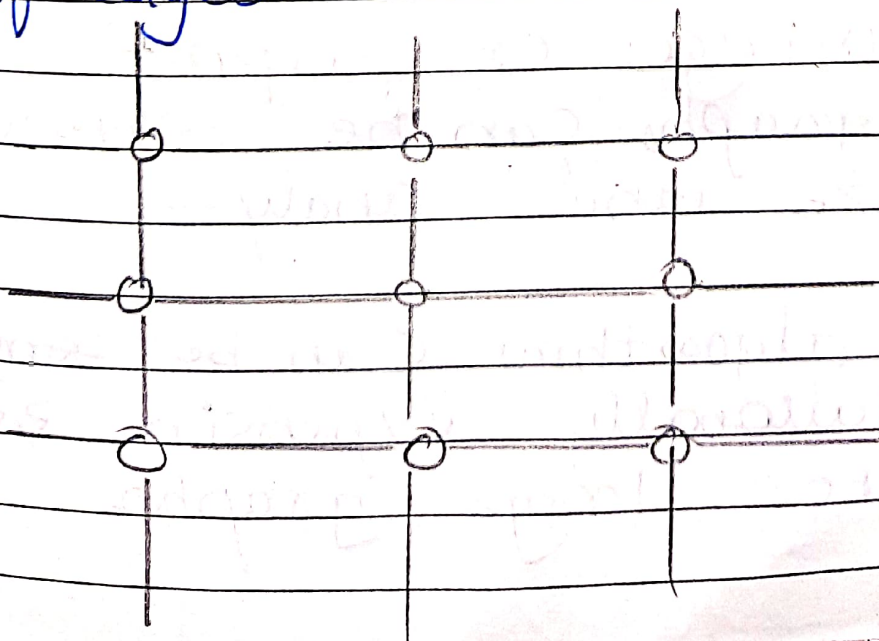
Ex:



18) Infinite graph:

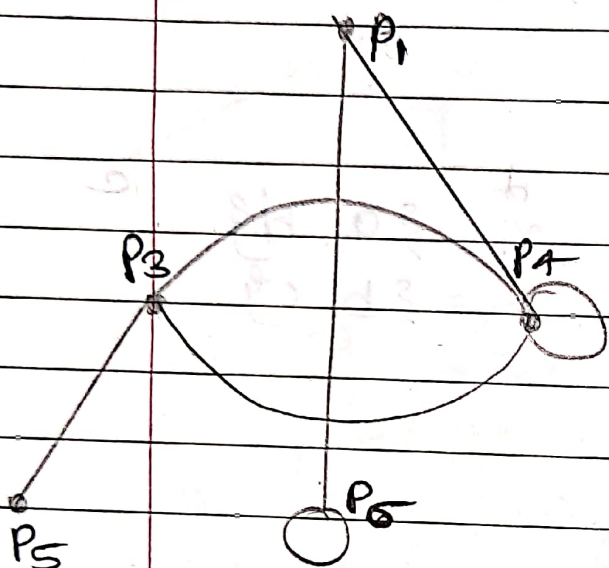
A graph is said to be infinite if it has an infinite number of vertices as well as an infinite number of edges.

Ex:



"Exercise."

1) Which of the following is true for given graph?



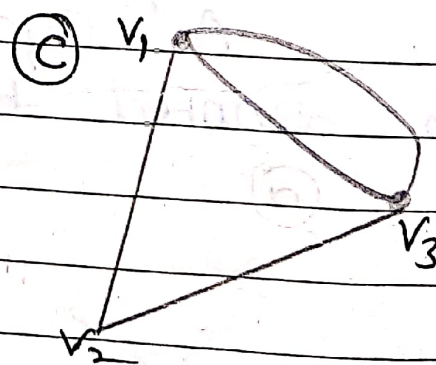
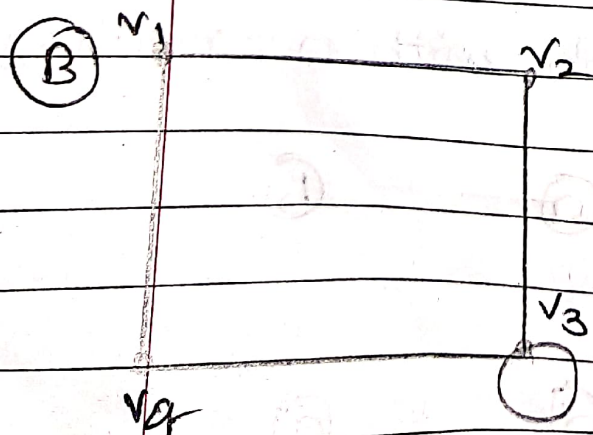
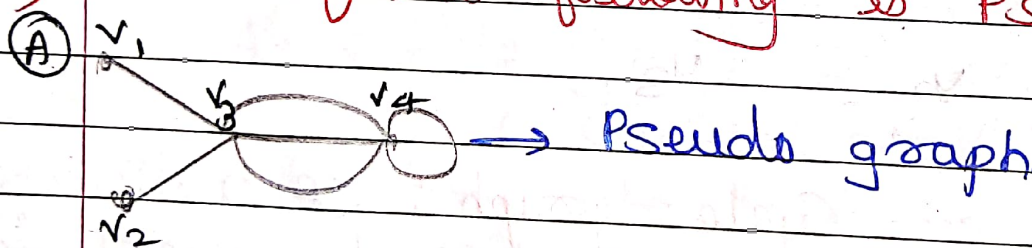
(A) $\deg(P_6) = 3$ ✓

(B) $\deg(P_5) = 2$ ✗

(C) $\deg(P_1) = 2$ ✓

(D) $\deg(P_2) = 1$ ✗

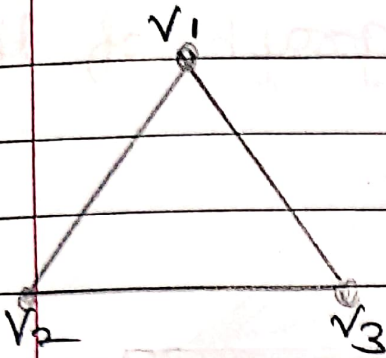
2) Which of the following is Pseudo graph?



(D) None of these

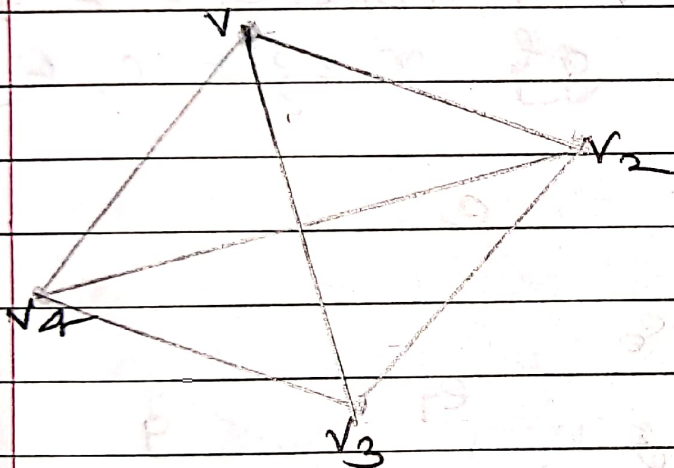
3) Draw a 2-Regular Graph

Soln $v = 2 + 1 = 3 = \{v_1, v_2, v_3\}$



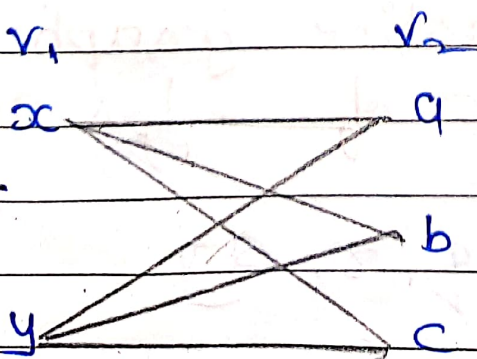
4) Draw a 3-Regular Graph

Soln $v = 3 + 1 = 4 = \{v_1, v_2, v_3, v_4\}$



5) Draw Bipartite or Bigraph $K_{2,3}$ and hence find number of edges and number of vertices.

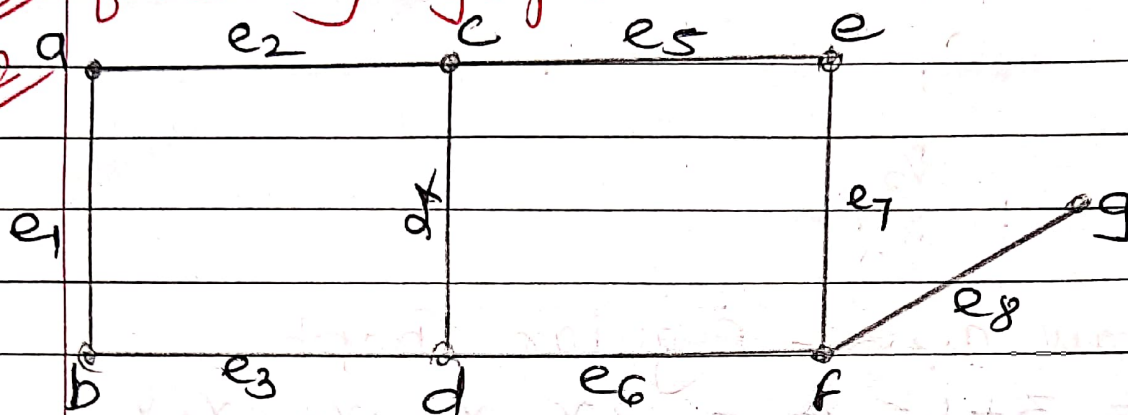
Soln $v_1 = \{x, y\}$ $v_2 = \{a, b, c\}$



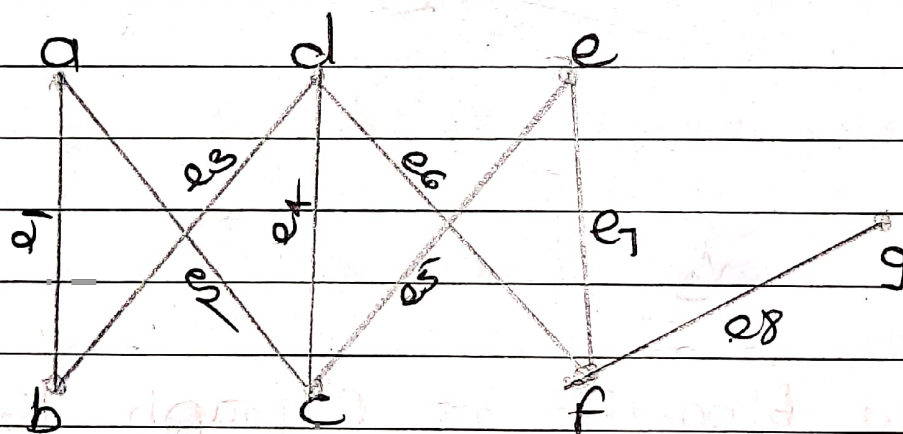
Number of edges $2 \times 3 = 6$

Number of vertices $2 + 3 = 6$

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5th Q) Define bipartite graph and draw corresponding bipartite graph of the following graph.



Soln: we draw as $V = \{a, b, c, d, e, f, g\}$
 $V_1 = \{a, d, e, g\}$ $V_2 = \{b, c, f\}$



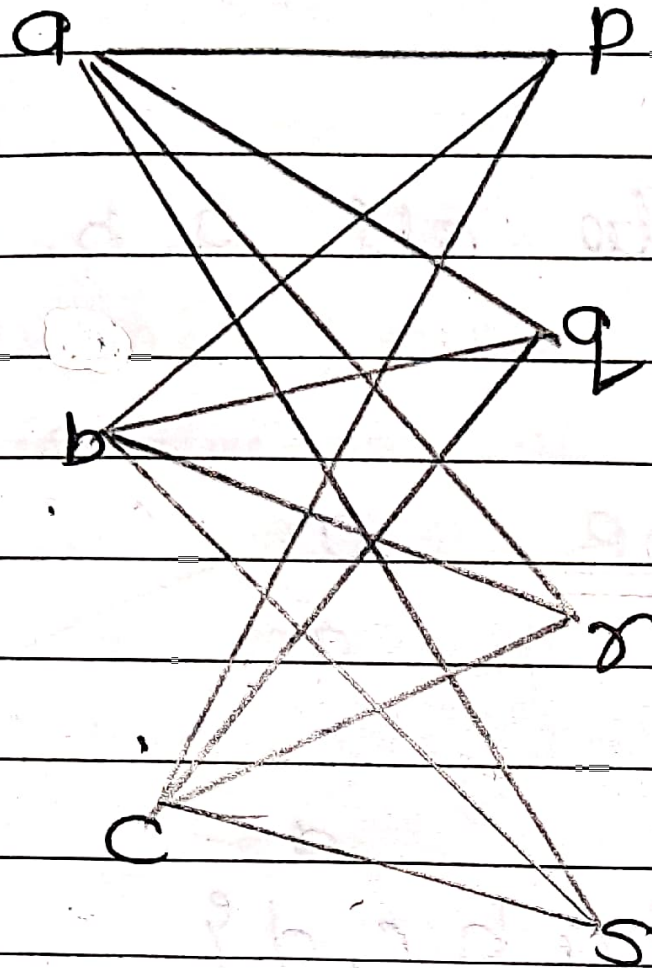
Number of edges = 8

Number of vertices (Node) = 7

Imp 7) Draw K_3 Bipartite graph and hence determine number of edges and number of vertices.

Soln: $V = \{a, b, c, p, q, r, s\}$

$$V_1 = \{a, b, c\} \quad V_2 = \{p, q, r, s\}$$

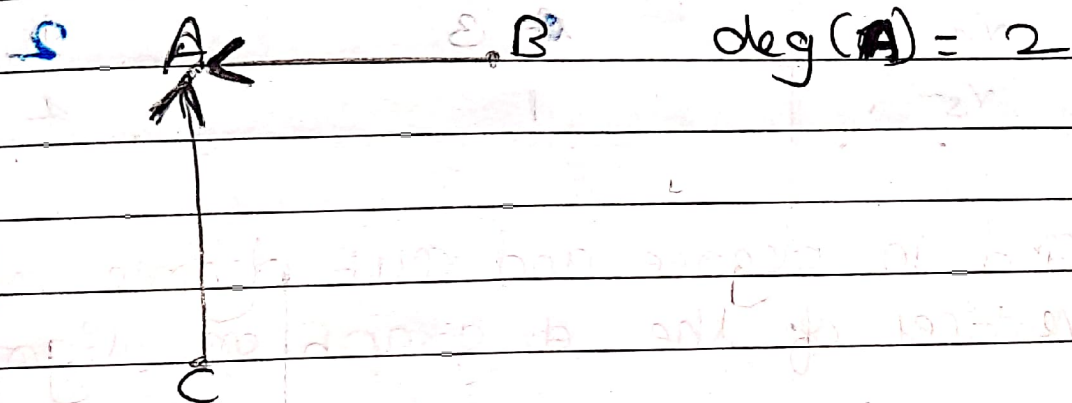


Number of edges = 3×4
 $= 12$

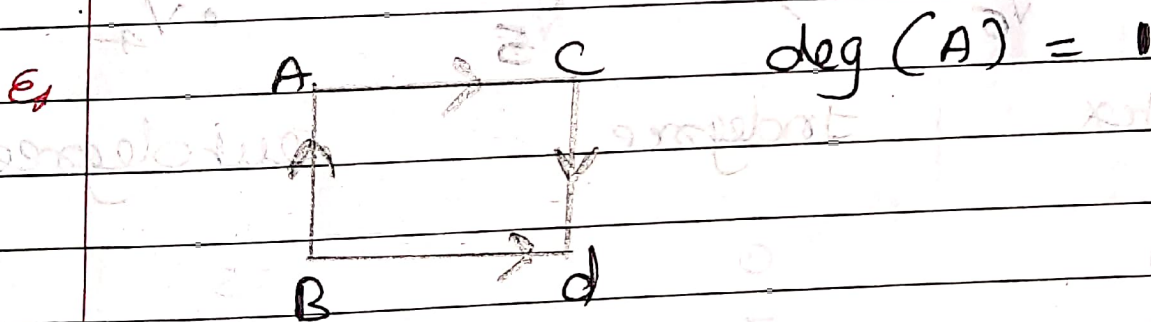
Number of vertices =
 $3 + 4 = 7$

Indegree and Outdegree of Graph

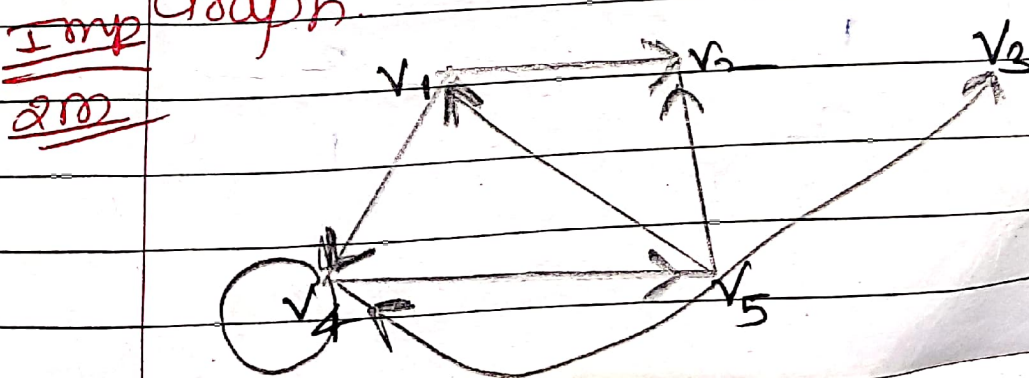
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Ex: Indegree of Graph: Indegree of vertex v is the number of edges which are coming into the vertex v .



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Ex: Outdegree of Graph: The number of edges directed out of a vertex in a directed graph

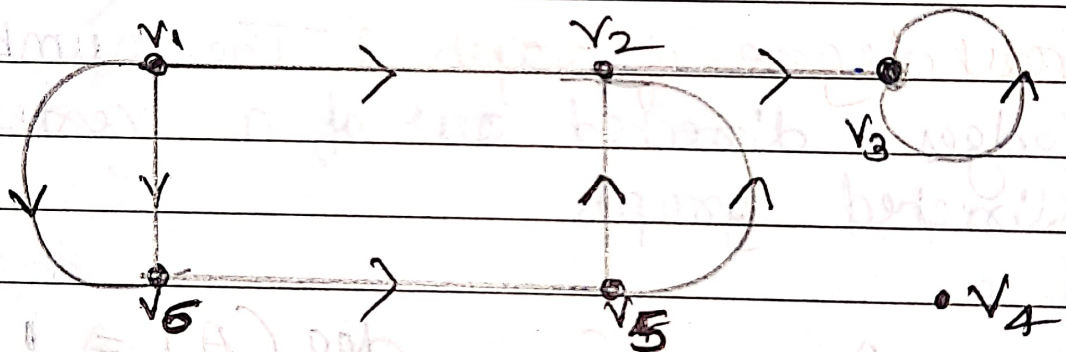


Ex 1) Find Indegree Outdegree for the Graph



Vertex	Indegree	out degree
v_1	1	2
v_2	2	0
v_3	1	0
v_4 (A)	3	2
v_5	1	4

2) Find in degree and out degree of the vertices of the graph or digraph



Vertex	Indegree	out degree
v_1	0	3
v_2	3	1
v_3	2	1
v_4	0	0
v_5	1	2
v_6	2	1

Representing Graph:

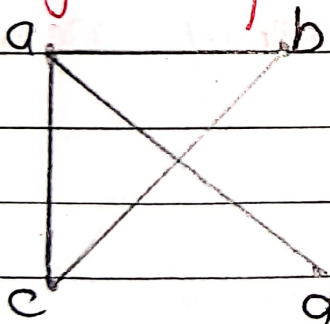
Graph can be represented in 2 way.

- 1) Adjacency Matrix
- 2) Adjacency list.

(i) Adjacency Matrix: It is a way to adjacency matrix of a graph $G = (V, E)$ is denoted and defined as

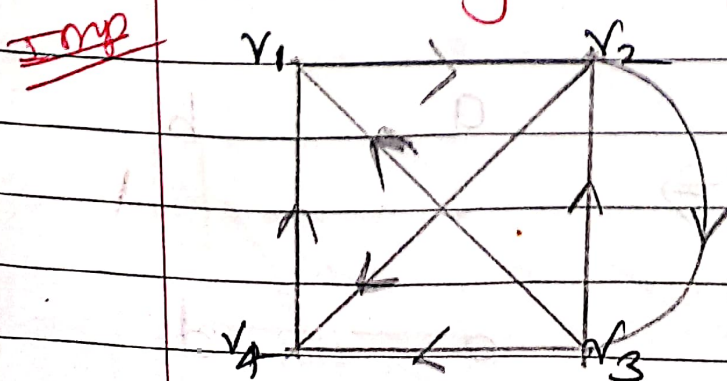
$$A = [a_{ij}] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

Ex: Represent following Graph with use of adjacency matrix.



	a	b	c	d
a	0	1	1	1
b	1	0	1	0
c	1	1	0	0
d	1	0	0	0

2) Write the matrix and find indegree and outdegree for the Graph below.



1) Matrix $V = \{v_1, v_2, v_3, v_4\}$

$$A = [a_{ij}] = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

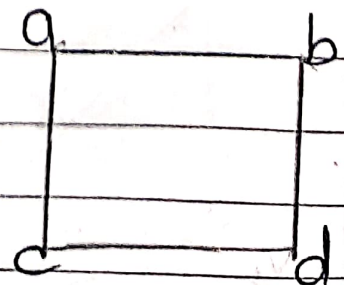
vertex	Indegree	outdegree
v_1	2	1
v_2	2	2
v_3	1	3
v_4	2	1

2) Draw a graph with Adjacency matrix.

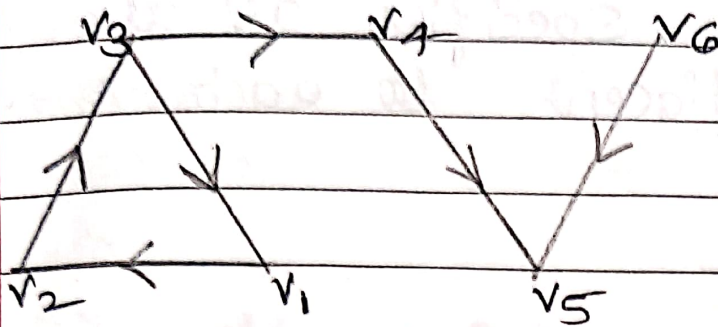
$$\begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Soln

$$\begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix} \Rightarrow$$



Q. Write Matrix and Indegree and outdegree for.



Soln) Matrix $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

← out going edge is 1 → no incoming edge is 0

$M_G =$
(Matrix Graph)

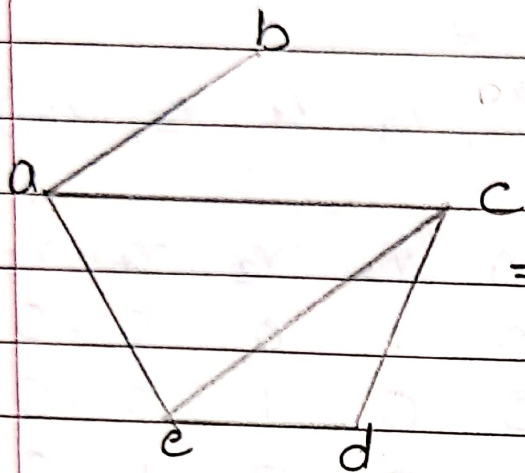
	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	0	0
v_2	0	0	1	0	0	0
v_3	1	0	0	1	0	0
v_4	0	0	0	0	1	0
v_5	0	0	0	0	0	0
v_6	0	0	0	0	1	0

Vertex	Indegree	outdegree
v_1	1	1
v_2	1	1
v_3	1	2
v_4	1	1
v_5	2	0
v_6	0	1

② Adjacency List :

It is a way to represent a graph without multiple edges. Specifies all the vertices that are adjacent to each vertex of a graph.

Ex

 $\Rightarrow$

vertex	Adjacent vertices
a	b, c, e
b	a
c	a, e, d
d	c, e
e	a, c, d

Graph Isomorphism : OR

Isomorphism of Graph : Definition :

Two graph G_1 and G_2 are isomorphic if

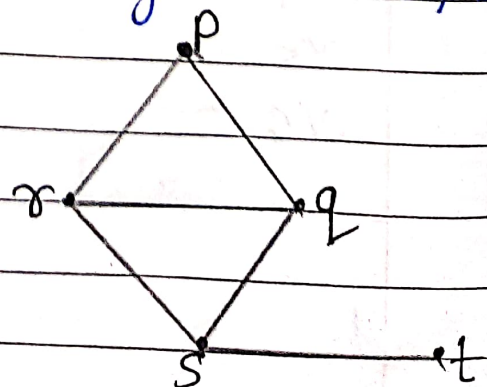
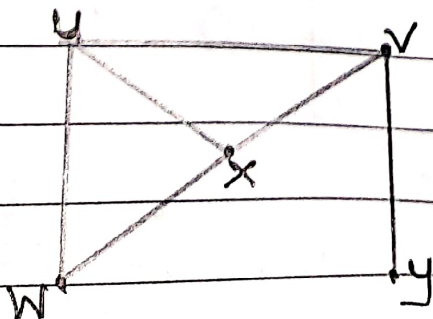
1) Number of vertices are same

2) Number of edges are same

3) An equal number of vertices with given degree

4) vertex correspondance and edge correspondance valid.

Ex: 1)



So/e Graph 1 :

Graph 2 :

$$V_1 = \{u, v, w, x, y\} \quad V_2 = \{p, q, r, s, t\}$$

$$d(u) = 3$$

$$d(p) = 2$$

$$d(v) = 3$$

$$d(q) = 3$$

$$d(w) = 2$$

$$d(r) = 3$$

$$d(x) = 3$$

$$d(s) = 3$$

$$d(y) = 1$$

$$d(t) = 1$$

$y \rightarrow t$ y Correspondence to t

$w \rightarrow p$

$v \rightarrow s$

$u \rightarrow q$

$x \rightarrow r$

y and w it is not the edge.

t and p it is not the edge.

~~Same~~

w and v it is not the edge

p and s it is not the edge.

v and u it is the edge

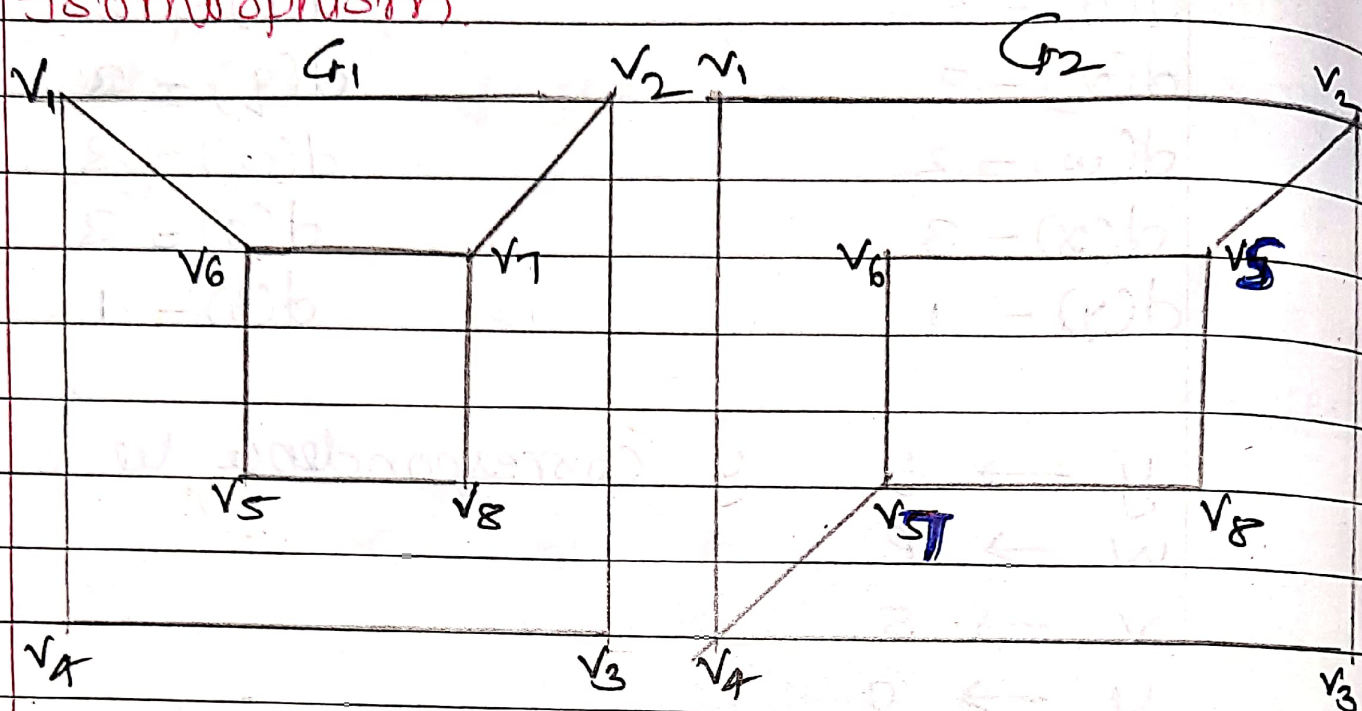
s and q it is the edge

u and x it is the edge

q and r it is the edge

Then above two graphs are isomorphic

2) Show that the graph G_1 and G_2 are isomorphism.



Solⁿ: Number of vertices of Graph 1 = 8
Number of vertices of Graph 2 = 8

Number of degree's.

Graph 1

$v_1 = 3$ $3 v_1 \rightarrow v_2$
 $v_2 = 3$ $2 v_3 \rightarrow v_8$
 $v_3 = 2$ $3 v_2 \rightarrow v_5$
 $v_4 = 2$ $2 v_4 \rightarrow v_3$
 $v_5 = 2$ $2 v_5 \rightarrow v_6$
 $v_6 = 3$ $3 v_6 \rightarrow v_7$
 $v_7 = 3$ $3 v_7 \rightarrow v_4$
 $v_8 = 2$ $2 v_8 \rightarrow v_1$

Graph 2

$v_2 = 3$
 $v_4 = 3$
 $v_3 = 2$
 $v_4 = 2$
 $v_6 = 2$
 $v_7 = 3$
 $v_5 = 3$
 $v_8 = 2$

Then above two graph it is a isomorphism.

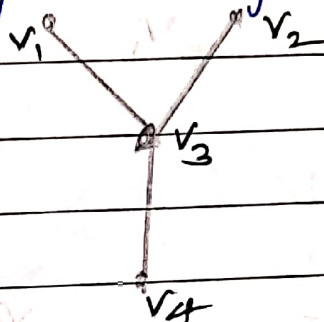
Tree:

A Graph or a structure with nodes or circles, that are connected by lines.

OR

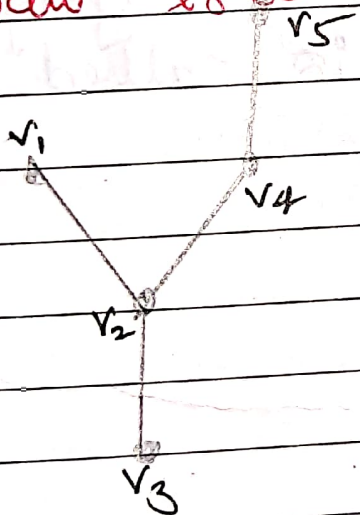
A Tree is an undirected graph in which any two vertices are connected by exactly one path.

ex 1)



2) Draw tree with 5 vertices.

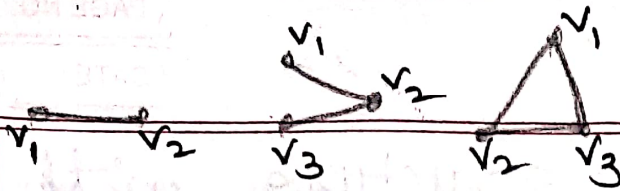
Soln.



Connectivity of a graph:

A graph is a connected graph if for each pair of vertices there exists at least one single path which join them.

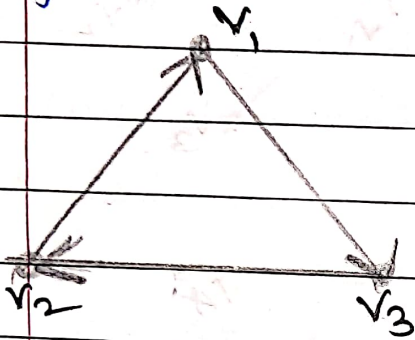
Ex:



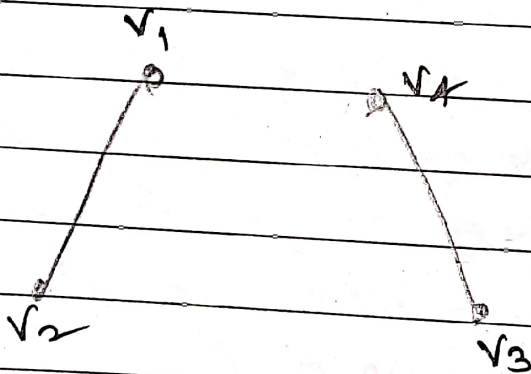
Connected Graph:

A graph G is called Connected if any ~~vertices~~ two vertices of the graph are connected by at least one path.

Ex:



Disconnected Graph: A graph that is not connected is called disconnected.

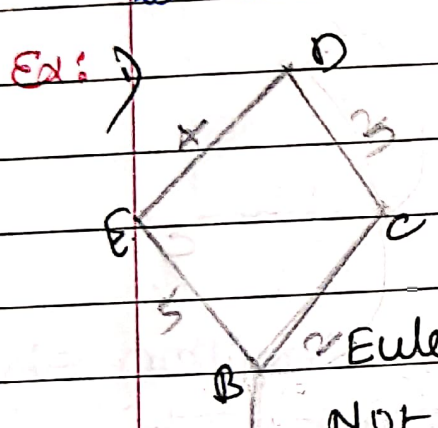


Euler Path:

9/12/22
 20/2/23
 15/1/23

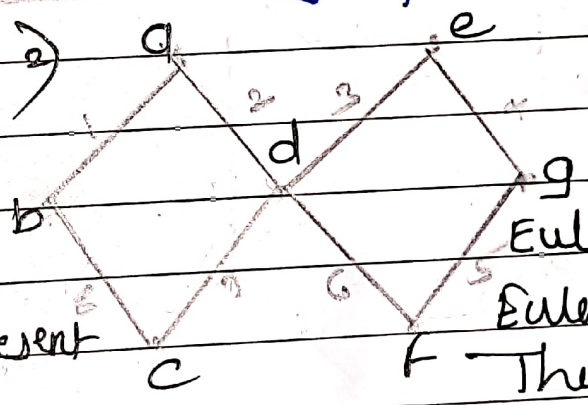
A path in a graph G is called Euler path if it includes every edge exactly once. Since the path contains every edge exactly once, it is also called Euler trail.

Euler Circuit: An Euler path that is a circuit is called Euler circuit. A graph which has an Eulerian circuit is called an Eulerian graph.

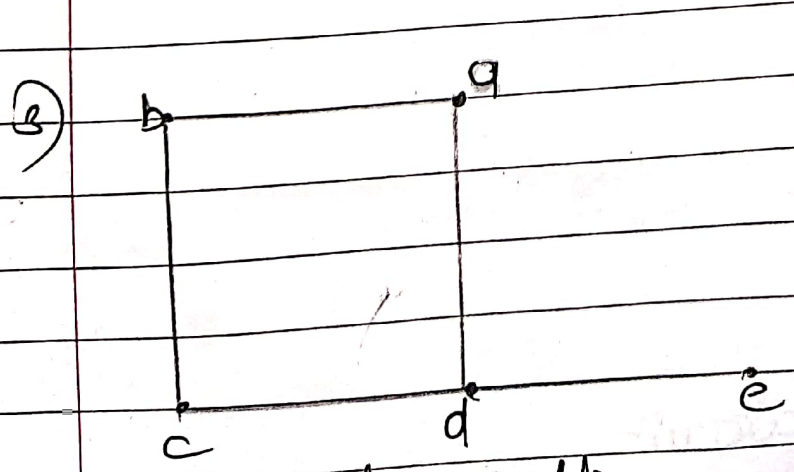


Euler path is present
 NOT Euler circuit

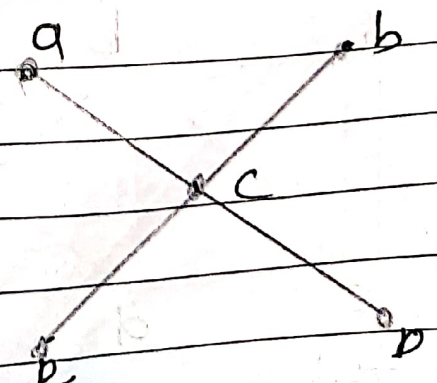
A Then it is not Euler path



Euler path
 Euler circuit
 Then it is Euler path



it is Euler path.



It is not Euler path

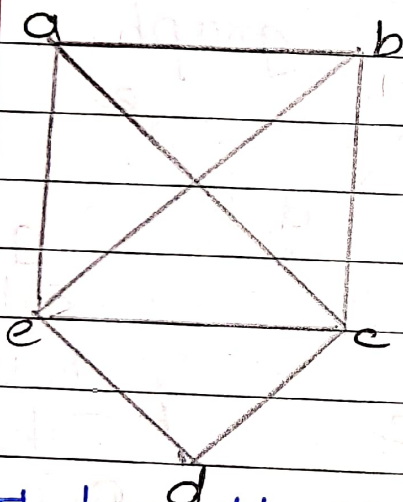
Hamilton Path:

It is a simple path passing through every vertex exactly once.

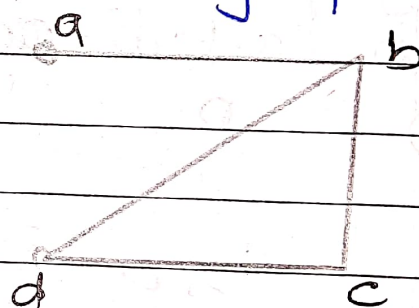
Hamilton Circuit: It is a simple circuit passing through every vertex exactly once.

Imp Hamilton Graph: A graph a Hamilton circuit is called Hamilton graph.

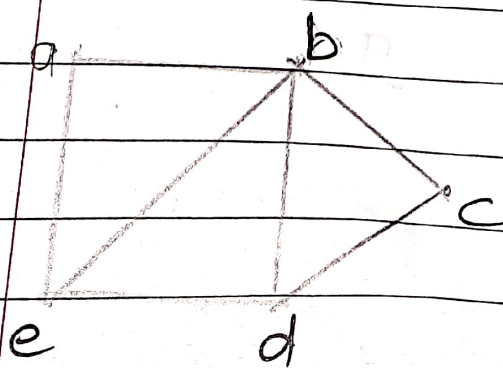
Ex:



It has Hamilton path and Hamilton circuit. a, b, c, d, e.



No Hamilton circuit. It has Hamilton path a, b, c, d.

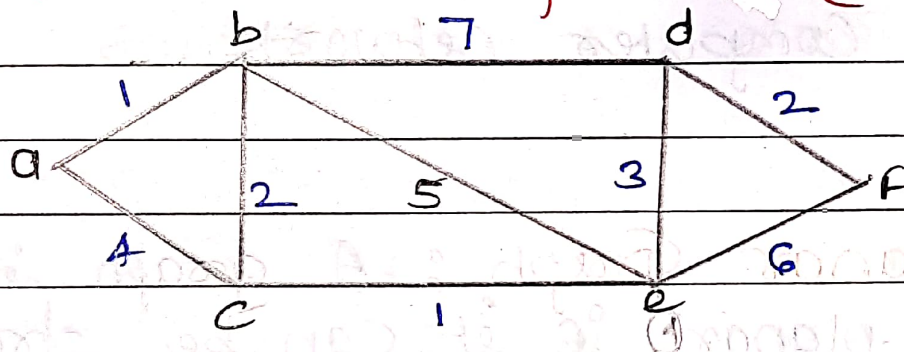


It has Hamilton circuit a, b, c, d, e, a.

Shortest-Path Problem :

Definition: Shortest Path Problem is the problem of finding a path between two vertices or nodes in a graph such that the sum of the weight of its constituent edges is minimized.

Ex:) find the shortest path (a - f)



vertices —	a	b	c	d	e	f
	0	1	4	∞	∞	∞
	0	1	3	8	6	∞
	0	1	3	8	4	∞
	0	1	3	7	4	10
	0	1	3	7	4	9

Shortest distance of between a - f is 9

Shortest path is a b c e d f

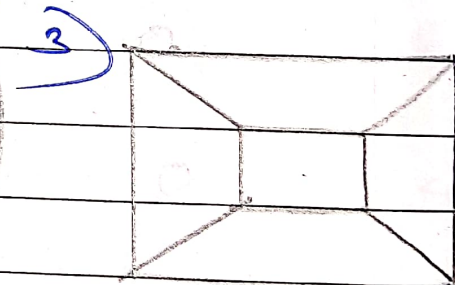
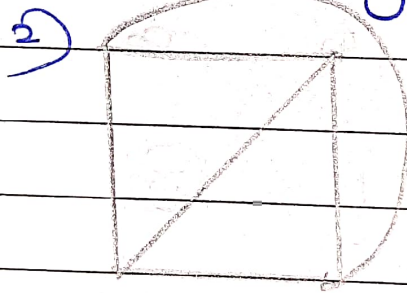
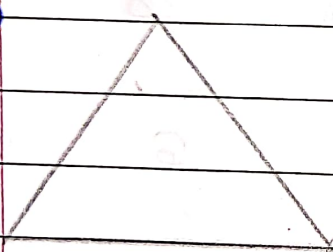
Application of Shortest

Path problem:

- 1) Digital mapping services in google.
- 2) In social networking
- 3) They are also important for road network, operations and logistics research.
- 4) In Computer network.

Planar Graph: A graph is said to be planar if it can be drawn in a plane so that no edge cross.

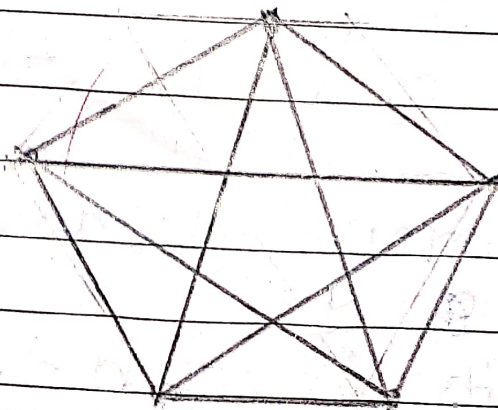
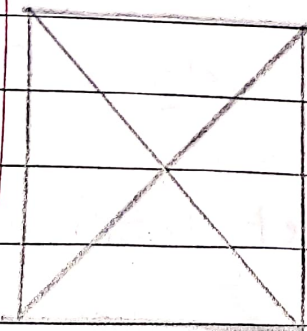
(I) Ex:



above all are Planar graph.

(II)

It is not a Planar graph



Graph Coloring:

Graph coloring is the procedure of assignment of colours to each vertex of a graph G such that no adjacent vertices should get same colour.

If G has n coloring then G is said to be n -colorable.

The chromatic number of a graph G is the minimum number of colours to color the vertices of the graph G and it is denoted by $\chi(G)$.

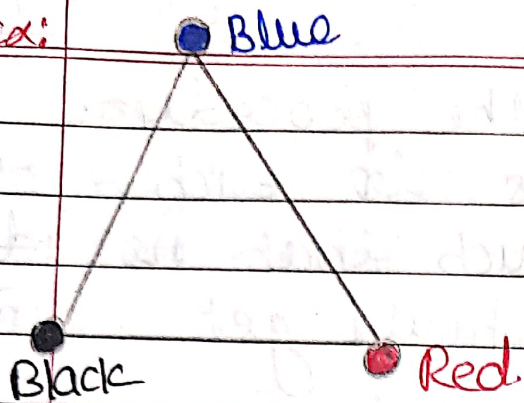
Step to required to colour a graph

- 1) Arrange the vertices of a graph in some order.
- 2) Choose the first vertex and colour it with the first colour.
- 3) Choose the next vertex and colour it with the different colour so that no two adjacent vertices should get same colour.

Exo -

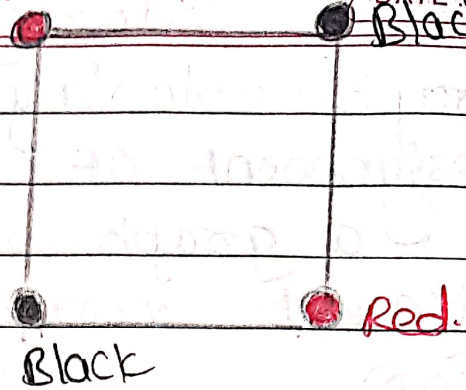
$$\chi(G) = 3$$

ex:



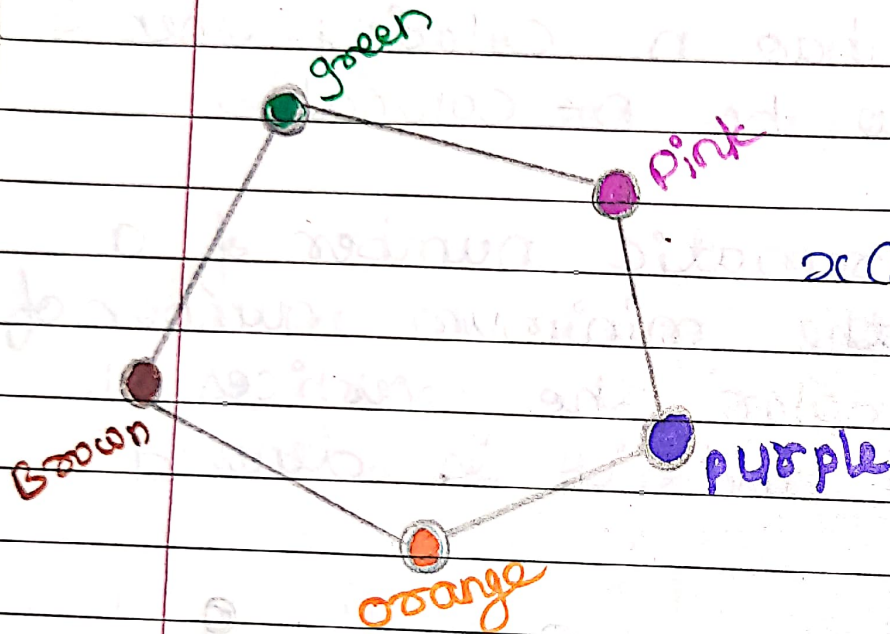
$$\chi(G) = 2$$

Red.



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$$\chi(G) = 5$$

Application of graph coloring:

- 1) Time table
- 2) Mobile Radio Frequency Assignment.
- 3) Register Allocation Allocation
- 4) Bipartite Graphs
- 5) Map coloring.

Advantages of Graph :

1. Graph can be used to model and analyze complex systems and relationships.
2. They are useful for visualizing and understanding data.
3. Graph algorithms are widely used in computer science and other fields such as social network analysis, logistics and transportation.
4. Graphs can be used to represent a wide range of data types, including social networks, road networks and the internet.

Disadvantages of graphs:

1. Large graphs can be difficult to visualize and analyze.
2. Graph algorithms can be very computationally expensive, especially for large graphs.

3. The interpretation of graph results can be subjective and may require domain specific knowledge.

4. Graphs can be susceptible to noise and outliers which can impact the accuracy of analysis results.