

Unit 3

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Induction And Recursion

The method of mathematical induction is based on a principle called the Induction principle.

This principle can be proved by using another principle known as the Well-ordering Principle.

Mathematical Induction Principle:

The Induction Principle state as follows.

⇒ Let $S(n)$ denote an open statement that involves a positive integer n . Suppose that the following condition hold.

- 1) $S(1)$ is true.
- 2) If whenever $S(k)$ is true for some particular, but arbitrarily chosen $k \in \mathbb{Z}^+$, then $S(k+1)$ is true.

Then $S(n)$ is true for all $n \in \mathbb{Z}^+$

Principle of Strong Induction:

Let P be a predicated on a set of +ve integers. If ① $P=0$ & for ② for all $n \in \mathbb{N}$ $P(0), P(1) \dots P(n+1)$ together are true then $P(m)$ is true $\forall m \in \mathbb{N}$.

Well-Ordering Principle:

The Well-Ordering Principle states as follows

Statement: Every non-empty subset of $\mathbb{Z}^+$ contains a smallest (least) element. An alternative way of stating this principle is that the set of all positive integers is well ordered.

Examples:

1) Prove by mathematical induction that for all positive integers $n \geq 1$

$$1+2+3+\dots+n = \frac{1}{2}n(n+1) \text{ OR } \frac{n(n+1)}{2}$$

Given

$$P(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Step (1): For $n=1$

$$\text{L.H.S.} = n = 1$$

$$\text{R.H.S.} = \frac{n(n+1)}{2}$$

$$= \frac{1(1+1)}{2}$$

$$\text{R.H.S.} = \frac{2}{2}$$

$$\text{R.H.S.} = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Step (2): for $n = k$.

Assume that $P(k)$ is true

$$\text{i.e. } P(k) = 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \rightarrow (1)$$

Step (3): for $n = k+1$

Adding $(k+1)$ term in eqn (1)

$$\text{i.e. } (1 + 2 + 3 + \dots + k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

L.H.S =

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k^2 + k}{2} + k + 1$$

$$= \frac{k^2 + k + 2(k+1)}{2}$$

$$= \frac{k^2 + k + 2k + 2}{2}$$

$$= \frac{1}{2} [k^2 + 3k + 2]$$

$$= \frac{1}{2} [k^2 + 2k + k + 2]$$

$$= \frac{1}{2} [k(k+2) + 1(k+2)]$$

$$= \frac{1}{2} [(k+2)(k+1)]$$

$$\text{L.H.S} = \frac{(k+1)(k+2)}{2}$$

$$R.H.S = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k^2 + k}{2} + (k+1)$$

$$= \frac{k^2 + k + 2(k+1)}{2}$$

$$= \frac{k^2 + k + 2k + 2}{2}$$

$$= \frac{1}{2} [k^2 + 3k + 2]$$

$$= \frac{1}{2} [k^2 + 2k + k + 2]$$

$$= \frac{1}{2} [k(k+2) + 1(k+2)]$$

$$= \frac{1}{2} [(k+2)(k+1)]$$

$$R.H.S = \frac{(k+1)(k+2)}{2}$$

$$L.H.S = R.H.S$$

$P(k+1)$ is true

Hence $P(n)$ is true for all $n \in \mathbb{N}$

② Prove that for each $n \in \mathbb{Z}^+$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

Solⁿ Given

$$P(n) = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

Step ①: for $n=1$

$$\text{L.H.S. } n^2 = 1^2 = 1$$

$$\text{R.H.S.} = \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2 \times 1+1)}{6}$$

$$= \frac{2(2+1)}{6} = \frac{2 \times 3}{6} = \frac{6}{6} = 1$$

$$\text{R.H.S.} = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$\therefore P(n)$ is true for $n=1$

Step (2) : for $n=k$

Assume that $P(k)$ is true.

$$\text{i.e. } 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \rightarrow (1)$$

Step (3) : for $n=k+1$

Adding $(k+1)^2$ term in eqn (1)

$$1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

L.H.S.

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1) \left[\frac{k(2k+1)}{6} + (k+1) \right]$$

$$= (k+1) \left[\frac{k(2k+1) + 6(k+1)}{6} \right]$$

$$= (k+1) \left[\frac{2k^2 + k + 6k + 6}{6} \right]$$

$$= \frac{1}{6} (k+1) [2k^2 + 7k + 6]$$

$$= \frac{1}{6} (k+1) [2k^2 + 4k + 3k + 6]$$

$$= \frac{1}{6} (k+1) [2k(k+2) + 3(k+2)]$$

$$= \frac{1}{6} (k+1) [(2k+3)(k+2)]$$

$$\text{L.H.S.} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\text{R.H.S.} = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1) \left[\frac{k(2k+1)}{6} + (k+1) \right]$$

$$= (k+1) \left[\frac{k(2k+1) + 6(k+1)}{6} \right]$$

$$= \frac{1}{6} (k+1) [2k^2 + k + 6k + 6]$$

$$= \frac{1}{6} (k+1) [2k^2 + 7k + 6]$$

$$= \frac{1}{6} (k+1) [2k^2 + 4k + 3k + 6]$$

$$= \frac{1}{6} (k+1) [2k(k+2) + 3(k+2)]$$

$$= \frac{1}{6} (k+1) [(2k+3)(k+2)]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

$\therefore P(k+1)$ is true.

Hence $P(n)$ is true for all $n \geq 1$

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Prove by mathematical induction that
 $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$
 for all integers $n \geq 1$

Soln: Given

$$P(n) = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Step (1): for $n=1$

$$\text{L.H.S.} = (2n-1)^2 = (2 \times 1 - 1)^2 = (2-1)^2 = 1^2 = 1$$

$$\text{L.H.S.} = 1$$

$$\text{R.H.S.} = \frac{n(2n-1)(2n+1)}{3}$$

$$= \frac{1(2 \times 1 - 1)(2 \times 1 + 1)}{3} = \frac{(2-1)(2+1)}{3}$$

$$= \frac{1 \times 3}{3} = \frac{3}{3} = 1$$

$$\text{R.H.S.} = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$\therefore P(n)$ is true for $n=1$

Step (2): for $n=k$

Assume that $P(k)$ is true

$$\text{i.e. } 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

Step (3): for $n = (2k+1)^2$ adding $\xrightarrow{3} \textcircled{1}$

$$\text{i.e. } 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$\text{L.H.S.} =$$

$$\frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= (2k+1) \left\{ \frac{1}{3} k(2k-1) + (2k+1) \right\}$$

$$= (2k+1) \left\{ \frac{k(2k-1) + 3(2k+1)}{3} \right\}$$

$$= (2k+1) \left\{ \frac{2k^2 - k + 6k + 3}{3} \right\}$$

$$= \frac{1}{3} (2k+1) \{ 2k^2 + 5k + 3 \}$$

$$= \frac{1}{3} (2k+1) \{ 2k^2 + 2k + 3k + 3 \}$$

$$= \frac{1}{3} (2k+1) \{ 2k(k+1) + 3(k+1) \}$$

$$= \frac{1}{3} (2k+1) \{ (2k+3)(k+1) \}$$

$$\text{L.H.S.} = \frac{1}{3} (2k+1)(k+1)(2k+3)$$

$$\text{R.H.S.} =$$

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$= (2k+1) \left\{ \frac{1}{3} k(2k-1) + (2k+1) \right\}$$

$$= (2k+1) \left\{ \frac{k(2k-1) + 3(2k+1)}{3} \right\}$$

$$= \frac{1}{3} (2k+1) \{ 2k^2 - k + 6k + 3 \}$$

$$= \frac{1}{3} (2k+1) \{ 2k^2 + 5k + 3 \}$$

$$= \frac{1}{3} (2k+1) \{ (2k+3) (k+1) \}$$

$$= \frac{1}{3} (2k+1) (k+1) (2k+3)$$

$$\therefore \text{L.H.S} = \text{R.H.S.}$$

$\therefore P(k+1)$ is true.

Hence $P(n)$ is true for all $n \geq 1$

4) If n is any positive integer prove that
 $1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$ or

$$\frac{n^2(n+1)^2}{4}$$

Soln Given Let

$$P(n) = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

Step (1): for $n=1$

$$\text{L.H.S} = n^3 = 1^3 = 1$$

$$\text{L.H.S} = 1$$

$$\text{R.H.S} = \frac{n^2(n+1)^2}{4}$$

$$\text{R.H.S} = \frac{1^2(1+1)^2}{4} = \frac{1(2)^2}{4} = \frac{4}{4} = 1$$

$$\text{R.H.S} = 1$$

$\therefore \text{L.H.S} = \text{R.H.S.}$

$\therefore P(n)$ is true for $n=1$

Step (2): for $n=k$

$$\text{i.e. } 1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4}$$

$\rightarrow \textcircled{1}$

Step (3): For $n = (k+1)$

Adding $(k+1)^3$ term in eqn (1)

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

L.H.S. =

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2 \left\{ \frac{1}{4} k^2 + (k+1) \right\}$$

$$= (k+1)^2 \left\{ \frac{k^2 + 4(k+1)}{4} \right\}$$

$$= \frac{1}{4} (k+1)^2 \{ k^2 + 4k + 4 \}$$

$$= \frac{1}{4} (k+1)^2 \{ (k+2)^2 \}$$

$$\text{L.H.S.} = \frac{(k+1)^2 (k+2)^2}{4}$$

$$\text{R.H.S.} = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2 \left\{ \frac{1}{4} k^2 + (k+1) \right\}$$

$$= \frac{1}{4} (k+1)^2 \{ k^2 + 4k + 4 \}$$

$$= \frac{1}{4} (k+1)^2 \{ (k+2)^2 \}$$

$$\text{R.H.S.} = \frac{(k+1)^2 (k+2)^2}{4}$$

L.H.S. = R.H.S. $\therefore P(n)$ is true for $n \geq 1$

5) If n is any positive integer

prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) =$

$$\frac{1}{3} n(n+1)(n+2)$$

using mathematical induction.

soln Given let

$$P(n) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$$

Step (1): For $n=1$

$$L.H.S = n(n+1) = 1(1+1) = 1(2) = 2$$

$$L.H.S = 2$$

$$R.H.S = \frac{1}{3} n(n+1)(n+2)$$

$$= \frac{1}{3} \times 1(1+1)(1+2) = \frac{1}{3} \times 2 \times 3$$

$$R.H.S = 2$$

$$L.H.S = R.H.S$$

Step (2): For $n=k$

$$i.e. 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) = \frac{1}{3} k(k+1)(k+2)$$

$$\text{Step (3): For } n = (k+1)(k+2) \rightarrow \textcircled{1}$$

Adding $(k+1)(k+2)$ term in eqn (1)

$$i.e. 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) + (k+1)(k+2) = \frac{1}{3} k(k+1)(k+2) + (k+1)(k+2)$$

$$L.H.S$$

$$= \frac{1}{3} k(k+1)(k+2) + (k+1)(k+2)$$

$$= (k+1)(k+2) \left\{ \frac{1}{3} k + 1 \right\}$$

$$(k+1)(k+2) \left\{ \frac{k+3}{3} \right\}$$

$$\text{L.H.S.} = \frac{1}{3} (k+1) (k+2) (k+3)$$

$$\text{R.H.S.} = \frac{1}{3} k(k+1)(k+2) + (k+1)(k+2)$$

$$= (k+1)(k+2) \left\{ \frac{1}{3} k + 1 \right\}$$

$$= (k+1)(k+2) \left\{ \frac{k+3}{3} \right\}$$

$$\text{R.H.S.} = \frac{1}{3} (k+1) (k+2) (k+3)$$

$$\text{L.H.S.} = \text{R.H.S.}$$

∴ $P(k+1)$ is true

Hence $P(n)$ is true for all $n \geq 1$

Recursive Definitions:

A sequence is an arrangement of element of a set in order as the first element, the second element and so on.

Ex: $a_1, a_2, a_3, \dots, a_n, \dots$

above is a sequence in which a_1 is the first element a_2 is the second element $\dots$ a_n is the n th element and so on.

Here the element a_n is called the General term, or the n th term.

of a sequence. And the sequence is denoted by $\{a_n\}$

In another sequence the first term will be denoted by a_0 . Then a_1 will be the second term a_2 will be the third term and a_{n-1} will be the n^{th} term. The sequence of this type will be denoted by a_{n-1} i.e. $a_0, a_1, a_2, a_3, \dots, a_{n-1}$.

For describing a sequence two methods are commonly used.

- ① The explicit method.
- ② Recursive method.

① The explicit Method :

In this method the general term of a sequence is explicitly indicated.

ex: The sequence of all positive even integers is explicitly defined as.

$$a_n = 2n, n \in \mathbb{Z}^+$$

② Recursive Method :

In this method first few terms of the sequence are explicitly indicated and the general term is obtained from the new term

which is already known

Ex: Fibonacci Series $F_n = F_{n-1} + F_{n-2}$
for $n \geq 2$

Fibonacci numbers:

The Fibonacci numbers are defined recursively by

$$F_0 = 0, F_1 = 1 \text{ and}$$

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$

Example:

1) Write the explicit formula for sequence - 4

Soln: Let Given - 4

$$a_n = (-1)^n (2)^{2n}$$

where $n = 1, 2, 3, 4, \dots$

$$a_1 = (-1)^1 (2)^{2 \times 1}$$

$$a_1 = -1 \times (2)^2$$

$$a_1 = -1 \times 4$$

$$a_1 = -4$$

$$n = 2$$

$$a_2 = (-1)^2 (2)^{2 \times 2}$$

$$a_2 = 1 \times (2)^4$$

$$a_2 = 1 \times 2 \times 2 \times 2 \times 2$$

$$a_2 = 16$$

$$n = 3$$

$$a_3 = (-1)^3 \times (2)^{2 \times 3}$$

$$a_3 = -1 \times (2)^6$$

$$a_3 = -1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

$$a_3 = -64$$

$$n = 4$$

$$a_4 = (-1)^4 \times (2)^{2 \times 4}$$

$$a_4 = 1 \times (2)^8$$

$$a_4 = 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

$$a_4 = 256$$

∴ The series is

$-4, -16, -64, 256, - \dots$

2) Find an explicit formula for a_n in each of the following cases.

i) $a_n = a_{n-1} - 2, a_1 = 0$

80) ii) Given $a_n = a_{n-1} - 2, a_1 = 0$

Where $n = 1, 2, 3, 4, \dots$

$$n = 2 \quad a_n = a_{n-1} - 2$$

$$a_2 = a_{2-1} - 2$$

$$a_2 = a_1 - 2$$

$$a_2 = 0 - 2$$

$$a_2 = -2$$

$$n = 3$$

$$a_3 = a_{3-1} - 2$$

$$a_3 = a_2 - 2$$

$$a_3 = -2 - 2$$

$$a_3 = -4$$

$$n = 4$$

$$a_i = a_{i-1} - 2$$

$$a_4 = a_3 - 2$$

$$a_4 = -4 - 2$$

$$a_4 = -6$$

1

$$a_n = -2(n-1)$$

$$a_n = -2(a_{n-1}) - a_{n-1} - 2$$

$$= -2, -4, -6, -8, -10, \dots$$

ii) $a_n = a_{n-1} + 3, a_1 = 2$

so/12 $a_n = a_{n-1} + 3, a_1 = 2$

$$n = 2$$

$$a_2 = a_{2-1} + 3$$

$$a_2 = a_1 + 3$$

$$a_2 = 2 + 3$$

$$a_2 = 5$$

$$n = 3$$

$$a_3 = a_{3-1} + 3$$

$$a_3 = a_2 + 3$$

$$a_3 = 5 + 3$$

$$a_3 = 8$$

$$n = 4$$

$$a_4 = a_{4-1} + 3$$

$$a_4 = a_3 + 3$$

$$a_4 = 8 + 3$$

$$a_4 = 11$$

↓

$$a_n = 3(n-1) + 2$$

$$a_n = \cancel{3(n-1) + 2} \quad a_{n-1} + 3$$

$$2, 5, 8, 11$$

Examples:

1) Write the recursive formula for sequence: 5, 10, 20, 40, -----

Soln Given Sequence = 5, 10, 20, 40, -----

$$a_1 = 5, a_2 = 10, a_3 = 20, a_4 = 40$$

$$a_2 = 2(a_1) = 2(5) = 10$$

$$a_3 = 2(a_2) = 2(10) = 20$$

$$a_4 = 2(a_3) = 2(20) = 40$$

$$a_5 = 2(a_4) = 2(40) = 80$$

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↓

$$a_n = 2(a_{n-1})$$

2) Write the Recursive formula for the sequence 3, 7, 11, 15, 19, 23, -----

Soln:- Given Sequence.

$$3, 7, 11, 15, 19, 23, -----$$

$$a_1 = 3, a_2 = 7, a_3 = 11, a_4 = 15,$$

$$a_5 = 19, a_6 = 23, \dots$$

$$a_2 = a_1 + 4 = 3 + 4 = 7$$

$$a_3 = a_2 + 4 = 7 + 4 = 11$$

$$a_4 = a_3 + 4 = 11 + 4 = 15$$

$$a_5 = a_4 + 4 = 15 + 4 = 19$$

$$a_6 = a_5 + 4 = 19 + 4 = 23$$

⋮

$$a_n = a_{n-1} + 4$$

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Soln Given Sequence $-4, 16, -64, 256, \dots$

$$a_1 = -4, a_2 = 16, a_3 = -64, a_4 = 256$$

$$a_2 = -4(a_1) = -4 \times (-4) = 16$$

$$a_3 = -4(a_2) = -4 \times (16) = -64$$

$$a_4 = -4(a_3) = -4 \times (-64) = 256$$

⋮

$$a_n = (-1)^n (2)^{2n}$$

Examples:

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1) The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Evaluate F_2, F_3, F_4, F_5 .

Soln Given $F_n = F_{n-1} + F_{n-2}$
 $F_0 = 0$ $F_1 = 1$

Then

$$F_n = F_{n-1} + F_{n-2}, \quad n = 2$$

$$F_2 = F_{2-1} + F_{2-2}$$

$$F_2 = F_1 + F_0$$

$$F_2 = 1 + 0$$

$$F_2 = 1$$

$$n = 3$$

$$F_3 = F_{3-1} + F_{3-2}$$

$$F_3 = F_2 + F_1$$

$$F_3 = 1 + 1$$

$$F_3 = 2$$

$$n = 4$$

$$F_4 = F_{4-1} + F_{4-2}$$

$$F_4 = F_3 + F_2$$

$$F_4 = 2 + 1$$

$$F_4 = 3$$

∴ $F_n = 0, 1, 1, 2, 3, \dots$

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Q) Write the recursive formula for the sequence 10, 13, 16, 19, ———

Sol: Given sequence 10, 13, 16, 19, ———

Here $a_1 = 10$, $a_2 = 13$, $a_3 = 16$, $a_4 = 19$

$$a_2 = a_1 + 3$$

$$a_2 = 10 + 3$$

$$a_2 = 13$$

$$a_3 = a_2 + 3$$

$$a_3 = 13 + 3$$

$$a_3 = 16$$

$$a_4 = a_3 + 3$$

$$a_4 = a_3 + 3$$

$$a_4 = 16 + 3$$

$$a_4 = 19$$

↓
↓
↓

$$a_n = a_{n-1} + 3$$

Lucas number: The Lucas numbers are defined recursively by

$$L_0 = 2, L_1 = 1$$

$$L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2$$

Example: -

- 1) The Lucas numbers are recursively defined by $L_0 = 2$ and $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Evaluate L_2 to L_5 .

Soln Given $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$
 $n = 2$

$$L_2 = L_{2-1} + L_{2-2}$$

$$L_2 = L_1 + L_0$$

$$L_2 = 1 + 2$$

$$L_2 = 3$$

$$n = 3$$

$$L_3 = L_{3-1} + L_{3-2}$$

$$L_3 = L_2 + L_1$$

$$L_3 = 3 + 1$$

$$L_3 = 4$$

$$n = 4$$

$$L_4 = L_{4-1} + L_{4-2}$$

$$L_4 = L_3 + L_2$$

$$L_4 = 4 + 3$$

$$L_4 = 7$$

$$n = 5$$

$$L_5 = L_{5-1} + L_{5-2}$$

$$L_5 = L_4 + L_3$$

$$L_5 = 7 + 4 = 11$$

Structural Induction:

Structural Induction is a method for proving that all the elements of a recursively defined data type have some property.

Application:

- 1) Structural Induction is a proof method that is used in mathematical logic.
- 2) Computer Science
- 3) graph theory
- 4) Some other mathematical fields

Recursive Algorithm: Definition:

An algorithm is called recursive if it solves a problem by reducing it to an instance of the same problem with smaller input.

- ex: 1) Generation of Factorial.
2) Fibonacci number Series.

Program Corrections:

A program is correct if it produces the correct output for every possible input.

Relation:

Cartesian Product:

Let A and B be two sets. Then the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the Cartesian Product. OR Cross Product OR Product Set of A and B . It is denoted by $A \times B$.

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

ex: If $A = \{1, 0, -1\}$ and $B = \{2, 3\}$ then $A \times B = ?$

sol: Given $A = \{1, 0, -1\}$ & $B = \{2, 3\}$ then $A \times B$

$$A \times B = \{ (1, 2), (1, 3), (0, 2), (0, 3), (-1, 2), (-1, 3) \}$$

And, /

$$B \times A = \{ (2, 1), (3, 1), (2, 0), (3, 0), (2, -1), (3, -1) \}$$

$$A \times B \neq B \times A$$

$A \times B$ is not the $B \times A$.

i.e. $A \times B \neq B \times A$

In general, $B \times A = \{ (b, a) \mid b \in B \text{ & } a \in A \}$

$A \times B$ can be defined even when $B = A$
 Thus we can have the product of a set A with itself and this product is defined by

$$A \times A = \{(a, b) \mid a \in A \text{ and } b \in A\}$$

The product $A \times A$ is also denoted by A^2

ex: If $A = \{1, 0, -1\}$ then A^2 OR $A \times A = ?$

sol: Given $A = \{1, 0, -1\}$

then

$$A^2 \text{ OR } A \times A = \{1, 0, -1\}, \{1, 0, -1\}$$

$$A \times A \text{ OR } A^2 = \{(1, 1), (1, 0), (1, -1), (0, 1), (0, -1),$$

$$(0, 0), (-1, -1), (-1, 1), (-1, 0), (-1, -1)\}$$

Examples:

1) Given $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 7, 8\}$
 write down the Cartesian product of sets A and B .

sol: Given $A = \{1, 2, 3, 4, 5\}$ $B = \{4, 5, 7, 8\}$

Then Cartesian Product $A \times B$ is

$$A \times B = \{(1, 4), (1, 5), (1, 7), (1, 8), (2, 4), (2, 5), (2, 7), (2, 8), (3, 4), (3, 5), (3, 7), (3, 8), (4, 4), (4, 5), (4, 7), (4, 8), (5, 4), (5, 5), (5, 7), (5, 8)\}$$

2) Let $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{4, 6\}$
 write down i) $B \times A$ ii) $A \times C$

iii) $(A \cup B) \times C$ iv) $(A \times B) \cup (B \times C)$ v) $A \cap (B \times C)$

soln: Let $A = \{1, 3, 5\}$ $B = \{2, 3\}$ $C = \{4, 6\}$

Then

$$i) B \times A = \{(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$$

$$ii) A \times C = \{(1, 4), (1, 6), (3, 4), (3, 6), (5, 4), (5, 6)\}$$

$$iii) (A \cup B) \times C$$

$$\Rightarrow (A \cup B) = \{1, 2, 3, 5\}, C = \{4, 6\}$$

Then

$$(A \cup B) \times C = \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4),$$

$$(3, 6), (5, 4), (5, 6)\}$$

$$iv) (A \times B) \cup (B \times C)$$

$$\Rightarrow A \times B = \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$$

$$B \times C = \{(2, 4), (2, 6), (3, 4), (3, 6)\}$$

then

$$(A \times B) \cup (B \times C) = \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3), (2, 4), (2, 6), (3, 4), (3, 6)\}$$

$$v) A \cap (B \times C)$$

$$\Rightarrow B \times C = \{(2, 4), (2, 6), (3, 4), (3, 6)\} \quad A = \{1, 3, 5\}$$

$$\therefore A \cap (B \times C) = \phi.$$

3) Find x and y in each of the cases

1) $(2x-3, 3y+1) = (5, 7)$

2) $(x+2, 4) = (5, 2x+y)$

So/1) $(2x-3, 3y+1) = (5, 7)$

$$2x-3 = 5$$

$$3y+1 = 7$$

$$2x = 5+3$$

$$3y = 7-1$$

$$2x = 8$$

$$3y = 6$$

$$x = \frac{8}{2}$$

$$y = \frac{6}{3}$$

$$x = 4$$

$$y = 2$$

2) $(x+2, 4) = (5, 2x+y)$

So/2 $(x+2, 4) = (5, 2x+y)$

$$x+2 = 5$$

$$4 = 2x+y$$

$$x = 5-2$$

$$2x+y = 4$$

$$x = 3$$

$$2 \times 3 + y = 4$$

$$6+y = 4$$

$$6+y = 4$$

$$y = 4-6$$

$$y = -2$$

$$x = 3$$

$$y = -2$$

Relation : Definition:

Relation is a subset of Cartesian product of two non-empty set A and B.

It is denoted by 'R'

$$\therefore R \subset A \times B$$

Examples:

1) Let $A = \{1, 3, 5, 7, 4\}$ Let R be the relation on 'A' defined by aRb iff $a < b$

Soln Given $A = \{1, 3, 5, 7, 4\}$

$$A \times A = \{1, 3, 5, 7, 4\} \times \{1, 3, 5, 7, 4\}$$

$$A \times A = \{(1, 3), (1, 1), (1, 5), (1, 7), (1, 4), (3, 1), (3, 3), (3, 5), (3, 7), (3, 4), (5, 1), (5, 3), (5, 5), (5, 7), (5, 4), (7, 1), (7, 3), (7, 5), (7, 7), (7, 4), (4, 1), (4, 3), (4, 5), (4, 7), (4, 4)\}$$

$$\text{Now } R = \{(a, b) \mid a < b\}$$

$$R = \{(1, 3), (1, 5), (1, 7), (1, 4), (3, 5), (3, 7), (3, 4), (5, 7)\}$$

2) Let $A = \{1, 2, 3, 4\}$ and let R be the relation on 'A' defined by aRb iff $a > b$

Soln Given $A = \{1, 2, 3, 4\}$

$$A \times A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$$

$$A \times A = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)\}$$

$$(4,3), (4,4)\}$$

$$R = \{(a,b) \mid a > b\}$$

$$R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$$

3) IF $A = \{1, 2, 3, 4\}$ Let R be a relation on A defined by aRb iff $a = b$

Solⁿ Given $A = \{1, 2, 3, 4\}$

$$A \times A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$$

$$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

$$\text{Now } R = \{(a,b) \mid a = b\}$$

$$R = \{(1,1), (2,2), (3,3), (4,4)\}$$

4) IF $A = \{1, 2, 3, 4\}$ and R be the relation defined by aRb iff " a divides b "

Solⁿ $A = \{1, 2, 3, 4\}$

$$A \times A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$$

$$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

$$\text{Now } R = \{(a,b) \mid a \text{ divides } b\}$$

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

Properties of Relation:

Following are the some important properties of relations.

Types of Relation:

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① Reflexive Relation:

A relation R on a set A is said to be reflexive if $(a, a) \in R$ for all $a \in A$.

ex: $\Rightarrow$ If $A = \{1, 2, 3, 4\}$ then the relation $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ find reflexive relation.

sol: $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
the $(1, 1), (2, 2), (3, 3), (4, 4)$ is reflexive $1, 2, 3, 4 \in A \forall 1, 1 \in R$.

2) If $A = \{a, b, c\}$ then the relation $R = \{(a, a), (b, b), (c, c)\}$ is it reflexive Relation.

sol: Given $A = \{a, b, c\}$, $R = \{(a, a), (b, b), (c, c)\}$
Then, the given relation is Reflexive. Because $R = \{(a, a), (b, b), (c, c)\}$ is Reflexive $(a, a) \in R$ $(b, b) \in R$ and $(c, c) \in R$

② Irreflexive Relation:

A relation R on a set A is said to be irreflexive relation if

Properties of Relation:

Following are the some important properties of relations.

① Reflexive Relation:

A relation R on a set A is said to be reflexive if $(a, a) \in R$ for all $a \in A$.

ex: 1) If $A = \{1, 2, 3, 4\}$ then the relation $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ find reflexive relation.

sol: 2 $A = \{1, 2, 3, 4\}$, $R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
The $(1, 1), (2, 2), (3, 3), (4, 4)$ is reflexive $1, 2, 3, 4 \in A \quad \forall \quad 1, 1 \in R$.

2) If $A = \{a, b, c\}$ then the relation $R = \{(a, a), (b, b), (c, c)\}$ is it reflexive Relation.

sol: 2 Given $A = \{a, b, c\}$, $R = \{(a, a), (b, b), (c, c)\}$.
Then the given relation is Reflexive. Because $R = \{(a, a), (b, b), (c, c)\}$ is Reflexive $(a, a) \in R$ $(b, b) \in R$ and $(c, c) \in R$.

② Irreflexive Relation:

A relation R on a set A is said to be irreflexive relation if

$(a, a) \notin R$ for any $a \in A$.

② Symmetric Relation:

A relation R on a set is said to be symmetric if $(a, b) \in R$ $\implies$ $(b, a) \in R \quad \forall (a, b) \in A$.

ex: If $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1)\}$ then find symmetric relation.

Solⁿ Given $A = \{1, 2, 3\}$ $R = \{(1, 1), (1, 2), (2, 1)\}$
Then the given relation is symmetric relation. Because $(1, 1), (1, 2), (2, 1) \in R$ or $(1, 2) \in R$ and $(2, 1) \in R \quad \forall 1, 2 \in A$.

④ Asymmetric Relation:

A relation which is not symmetric is called an Asymmetric Relation.

⑤ Antisymmetric Relation:

A relation R on a set A is said to be antisymmetric. If whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

⑥ Transitive Relation:

A relation R on a set A is said to be transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

for all $(\forall) a, b, c \in A$.

ex: If $A = \{1, 2, 3\}$ and the relation $R = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$ then is it Transitive relation.

sol: Given $A = \{1, 2, 3\}$ $R = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$
 Then it is transitive relation
 $(1, 2) \in R, (2, 3) \in R$ and $(1, 3) \in R$
 $\forall 1, 2, 3 \in A$.

Examples :-

1) Let $A = \{1, 2, 3\}$ and the relation $R = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$ Determine whether the relation is Reflexive, Symmetric and Transitive.

sol: Given $A = \{1, 2, 3\}$ and Relation $R = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$.

1) Reflexive: $(1, 1), (2, 2) \notin R$ then the given relation is not Reflexive.

2) Symmetric: $(1, 2), (2, 1) \in R$ then it is symmetric relation.

3) Transitive: $(1, 2) \in R, (2, 3) \in R$ but $(1, 3) \notin R$. Then it is not Transitive relation.

2) Let $A = \{1, 2, 3\}$ and the relation

$$R = \{(1, 1), (1, 2), (2, 2), (3, 3), (2, 3), (1, 3)\}$$

Determine whether the relation is Reflexive, Symmetric and Transitive.

soln Given $A = \{1, 2, 3\}$ and Relation

$$R = \{(1, 1), (2, 2), (3, 3), (2, 3), (1, 3), (1, 2)\}$$

1) Reflexive: $(1, 1) \in R$, $(2, 2) \in R$ & $(3, 3) \in R$
 Thus it is Reflexive relation

2) Symmetric: $(1, 2) \in R$ & $(2, 1) \notin R$
 Thus it is not Symmetric relation

3) Transitive: $(1, 2) \in R$, $(2, 3) \in R$ and $(1, 3) \in R$ Thus it is Transitive relation.

Equivalence Relation:

~~Imp~~
~~Def~~
~~Def~~
 A relation R on a set A is said to be an equivalence relation on A if R is reflexive, Symmetric and Transitive.

Examples:

Imp 1) Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (3, 3), (4, 4)\}$ Show that is an Equivalence Relation.

soln Given $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (3, 3), (4, 4)\}$.

1) Reflexive: $(1, 1) \in R, (2, 2) \in R, (3, 3) \in R, (4, 4) \in R \forall 1, 2, 3, 4 \in A$.

Then it is a Reflexive relation.

2) Symmetric: $(1, 2) \in R, (2, 1) \in R, (1, 3) \in R, (3, 1) \in R \forall 1, 2, 3, 4 \in A$.
Then it is a Symmetric relation.

3) Transitive: $(1, 2) \in R, (2, 3) \in R$ and $(1, 3) \in R \forall 1, 2, 3, 4 \in A$. Then it is a Transitive relation.

Then the above relation R is an Equivalence Relation.

2) Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (a, b), (b, a), (b, c), (a, c)\}$ Show that is an Equivalence Relation.

soln Given $A = \{a, b, c\}$ $R = \{(a, a), (b, b), (a, b), (b, a), (b, c), (a, c)\}$

1) Reflexive: $(a, a) \in R, (b, b) \in R \forall a, b \in A$
Then it is Reflexive relation.

2) Symmetric: $(a, b) \in R, (b, a) \in R \forall a, b \in A$ Then it is Symmetric relation.

3) Transitive: $(a, b) \in R, (b, c) \in R$ and $(a, c) \in R$ then $\forall a, b, c \in A$ then it is Transitive relation.

Then the above relation R is
an Equivalence Relation.

Equivalence Relation and partition:
A relation R on a set A is an equivalence relation if it is reflexive, symmetric and transitive. If R is an equivalence relation on the set A , its equivalence classes form a partition of A .

Partition:

Let A be a non empty set. Suppose there exist nonempty subsets $A_1, A_2, A_3, \dots, A_k$ of A such that the following two conditions hold.

A) A is the union of $A_1, A_2, \dots, A_k$
 i.e. $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k$.

B) Any two of the subsets $A_1, A_2, A_3, \dots, A_k$ are disjoint

i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$

Then the set $P = \{A_1, A_2, A_3, \dots, A_k\}$ is called a partition of A .

Ex: List all partition of $A = \{1, 2, 3\}$

Soln: $A = \{1, 2, 3\}$

Then the ~~part~~ partition of A is
 $= \{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\},$
 $\{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}$

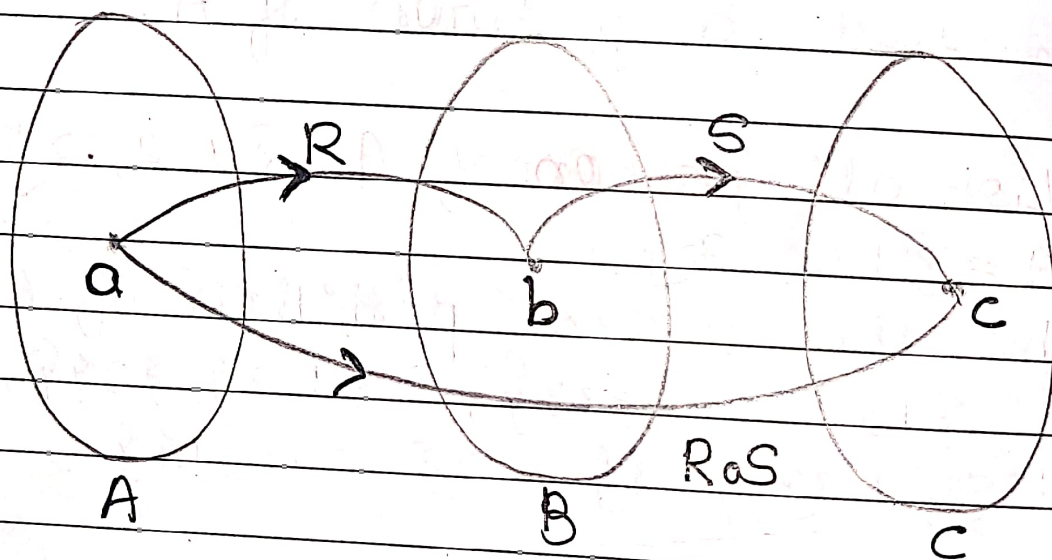
Composition of Relation:

Consider a relation R from a set A to a set B and a relation S from the set B to a set C . With these relation in hand we can define a new relation called the product or the composition of R and S .

from the set A to the set C .
And this relation is denoted
by $R \circ S$.

∴ $R \circ S = \{(a, c) \mid a \in A, c \in C \text{ and there exist } b \in B \text{ with } (a, b) \in R \text{ and } (b, c) \in S\}$
We observe that

$$R \subseteq (A \times B), S \subseteq (B \times C), \text{ \& } R \circ S \subseteq (A \times C)$$



Examples:

Imp

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1) Let $A = \{a, b, c, d\}$ where $R = \{(a, a), (a, d), (b, d)\}$ and $S = \{(a, d), (b, c), (b, d), (c, d)\}$ find $R \circ S$, $S \circ R$ and S^2

Solⁿ

$A = \{a, b, c, d\}$, $R = \{(a, a), (a, d), (b, d)\}$
 $S = \{(a, d), (b, c), (b, d), (c, d)\}$

$$R \circ S = \{(a, d), f\}$$

$$S = \{(a, d), (b, c), (b, d), (c, d)\}$$

$$R = \{(a, a), (a, d), (b, d)\}$$

$$S \circ R = \{ \}$$

$$R^2 = R \circ R = \{(a, a), (a, d), (b, d)\} \\ \{(a, a), (a, d), (b, d)\}$$

$$R^2 = \{(a, d), (a, a)\}$$

$$S^2 = S \circ S = \{(a, d), (b, c), (b, d), (c, d)\} \\ = \{(a, d), (b, c), (b, d), (c, d)\}$$

$$S^2 = \{bd\}$$

- 2) Let $A = \{1, 2, 3, 4\}$ $B = \{x, y, z, w\}$
 $C = \{5, 6, 7\}$, $R = \{(1, x), (2, x), (3, y), (3, z)\}$
 $S = \{(w, 5), (x, 6)\}$ and $T = \{(w, 5), (w, 6)\}$
 Find i) $R \circ S$ ii) $R \circ T$ iii) $S \circ T$.

80/2: Given $A = \{1, 2, 3, 4\}$ $B = \{x, y, z, w\}$
 $C = \{5, 6, 7\}$

i) $R \circ S = R = \{(1, x), (2, x), (3, y), (3, z)\}$
 $S = \{(w, 5), (x, 6)\}$

$$R \circ S = \{(1, 4), (2, 4)\}$$

ii) $R \circ T$

$$\Rightarrow R = \{(1, x), (2, x), (3, y), (3, z)\}$$

$$T = \{(w, 5), (w, 6)\}$$

∴

$$R \circ T = \{\}$$

iii) $S \circ T$

$$\Rightarrow S = \{(w, 5), (x, 6)\}$$

$$T = \{(w, 5), (w, 6)\}$$

$$S \circ T = \{\}$$

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3) Let $A = \{1, 2, 3, 4\}$ $R = \{(1, 1), (1, 2), (2, 3), (3, 4)\}$ $S = \{(3, 1), (4, 4), (2, 4), (1, 4)\}$ be a relation on A . Determine the relation $R \circ S$, $S \circ R$, R^2 and S^2

Sol: Given $A = \{1, 2, 3, 4\}$

$R \circ S$

$$\Rightarrow R = \{(1, 1), (1, 2), (2, 3), (3, 4)\}$$

$$S = \{(3, 1), (4, 4), (2, 4), (1, 4)\}$$

$$R \circ S = \{(1, 4), (2, 1), (3, 4)\}$$

$S \circ R$

$$\Rightarrow S = \{(3,1), (4,4), (2,4), (1,4)\}$$

$$R = \{(1,1), (1,2), (2,3), (3,4)\}$$

$$S \circ R = \{(3,1), (3,2)\}$$

$$R^2 = R \circ R = \{(1,1), (1,2), (2,3), (3,4)\}$$

$$= \{(1,1), (1,2), (2,3), (3,4)\}$$

$$R^2 = \{(1,1), (1,2), (1,3), (2,4)\}$$

$$S^2 = S \circ S = \{(3,1), (4,4), (2,4), (1,4)\}$$

$$= \{(3,1), (4,4), (2,4), (1,4)\}$$

$$S^2 = \{(3,4), (4,4), (2,4), (1,4)\}$$

4) let $R = \{(1,2), (3,4), (2,2)\}$

$$S = \{(4,2), (2,5), (3,1), (1,3)\}$$

Find $R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $R \circ R$, $S \circ S$

Solⁿ $R = \{(1,2), (3,4), (2,2)\}$

$$S = \{(4,2), (2,5), (3,1), (1,3)\}$$

$$R \circ S = \{(1,5), (3,2), (2,5)\}$$

 $S \circ R$

$$\Rightarrow S = \{(4,2), (2,5), (3,1), (1,3)\}$$

$$R = \{(1,2), (3,4), (2,2)\}$$

$$S \circ R = \{(1, 2), (3, 2), (1, 4)\}$$

$$R \circ (S \circ R)$$

$$\Rightarrow R = \{(1, 2), (3, 4), (2, 2)\}$$

$$S \circ R = \{(4, 2), (3, 2), (1, 4)\}$$

$$R \circ (S \circ R) = \{(3, 2)\}$$

$$R^2 = R \circ R = \{(1, 2), (3, 4), (2, 2)\}$$

$$= \{(1, 2), (3, 4), (2, 2)\}$$

$$R \circ R = R^2 = \{(1, 2), (2, 2)\}$$

$$S \circ S = S^2 = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$$

$$= \{(4, 2), (2, 5), (3, 1), (1, 3)\}$$

$$S^2 = \{(4, 5), (3, 3), (1, 1)\}$$

Closeure Operation on Relation :

defⁿ: The closeure of a Relation R with respect to property p is the relation obtained by adding the minimum number of ordered pairs to R obtain property p .

1) Reflexive closeure 2) Symmetric closeure

③ Transitive closure.

Operation On Relation :

① Union of a Relation :

Union of two relations R and S .
It is denoted by $R \cup S$.

And defined by

$$R \cup S = \{(a, b) \in R \cup S \text{ iff } (a, b) \in R \text{ or } (a, b) \in S\}$$

② Intersection of relation :

Intersection of two relation R and S . It is denoted by $R \cap S$.

And defined by

$$R \cap S = \{(a, b) \in R \text{ and } (a, b) \in S\}$$

③ Complement of a Relation :

Complement of relation is denoted as $\bar{R}$ and defined by

$$(a, b) \in \bar{R} \text{ iff } (a, b) \notin R$$

④ Converse of a Relation :

Converse of a relation is denoted as R^c and defined by

$$(a, b) \in R^c \text{ iff } (b, a) \in R$$

Examples:

1) Consider the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ and the relation $R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$ and $S = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$ from A to B , determine $\overline{R}$, $\overline{S}$, $R \cup S$, $R \cap S$, R^c and S^c

Sol: Given $A = \{a, b, c\}$ $B = \{1, 2, 3\}$
We first note that

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$$

in

out

$\overline{R}$

$$\Rightarrow \overline{R} = \text{complement of } R \text{ in } A \times B = (A \times B) - R$$

$$= \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\} - \{(a, 1), (b, 1), (c, 2), (c, 3)\}$$

$$= \overline{R} = \{(a, 2), (a, 3), (b, 2), (b, 3), (c, 1)\}$$

$\overline{S}$

$$\Rightarrow \overline{S} = (A \times B) - S$$

$$= \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\} - \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

$$S = \{(a, 3), (b, 3), (c, 1), (c, 2), (c, 3)\}$$

RVS

$$\Rightarrow RVS = \{(a, 1), (b, 1), (c, 2), (c, 3)\} \cup \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

$$= RVS = \{(a, 1), (b, 1), (c, 2), (c, 3), (a, 2), (b, 2)\}$$

RNS

$$\Rightarrow RNS = \{(a, 1), (b, 1), (c, 2), (c, 3)\} \cap \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

$$RNS = \{(a, 1), (b, 1)\}$$

 R^c

$$\Rightarrow R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$$

$$R^c = \{(1, a), (1, b), (2, c), (3, c)\}$$

 S^c

$$\Rightarrow S = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$$

$$S^c = \{(1, a), (2, a), (1, b), (2, b)\}$$

2) Let $A=B=\{1,2,3\}$ and $R=\{(1,1), (1,2), (2,3), (3,1)\}$, $S=\{(2,1), (3,1), (3,2), (3,3)\}$
 Compute $\bar{R}$, $R \cap S$, $R \cup S$ and S^c

Soln Let $A=B=\{1,2,3\}$ $R=\{(1,1), (1,2), (2,3), (3,1)\}$
 $S=\{(2,1), (3,1), (3,2), (3,3)\}$

$\bar{R}$

$$\Rightarrow A \times B = \{1,2,3\} \times \{1,2,3\}$$

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

$$\bar{R} = (A \times B) - R$$

$$= \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\} - \{(1,1), (1,2), (2,3), (3,1)\}$$

$$\bar{R} = \{(1,3), (2,1), (2,2), (3,2), (3,3)\}$$

$$R \cap S =$$

$$\Rightarrow \{(1,1), (1,2), (2,3), (3,1)\} \cap \{(2,1), (3,1), (3,2), (3,3)\}$$

$$R \cap S = \{(3,1)\}$$

$$R \cup S$$

$$\Rightarrow \{(1,1), (1,2), (2,3), (3,1)\} \cup \{(2,1), (3,1), (3,2), (3,3)\}$$

$$R \cup S = \{(1,1), (1,2), (2,3), (3,1), (2,1), (3,2), (3,3)\}$$

S^c

$$\Rightarrow S = \{(2,1), (3,1), (3,2), (3,3)\}$$

$$S^c = \{(1,2), (1,3), (2,3), (3,3)\}$$

3) Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relation R and S from A to B are represented by the following matrices determine the relation $\bar{R}$, $R \cup S$, $R \cap S$ and S^c and their matrix representation

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solⁿ By examining M_R and M_S we note that

$$R = \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3)\}$$

$$S = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,2), (3,4)\}$$

 $\bar{R}$

$$= A \times B = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$$

$$\bar{R} = (A \times B) - R$$

$$\bar{R} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\} - \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3)\}$$

$$\bar{R} = \{(1,2), (1,4), (2,1), (2,2), (2,3), (3,4)\}$$

RVS.

$$= \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3)\} \cup \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,2), (3,4)\}$$

$$R \cup S = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,1), (3,2), (3,3), (3,4)\}$$

R ∩ S =

$$= \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3)\} \cap \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,2), (3,4)\}$$

$$R \cap S = \{(1,1), (1,3), (2,4), (3,2)\}$$

 S^c

$$S = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,2), (3,4)\}$$

$$S^c = \{(1,1), (2,1), (3,1), (4,1), (4,2), (2,3), (4,3)\}$$

The matrix representation of the above relations are as given below.

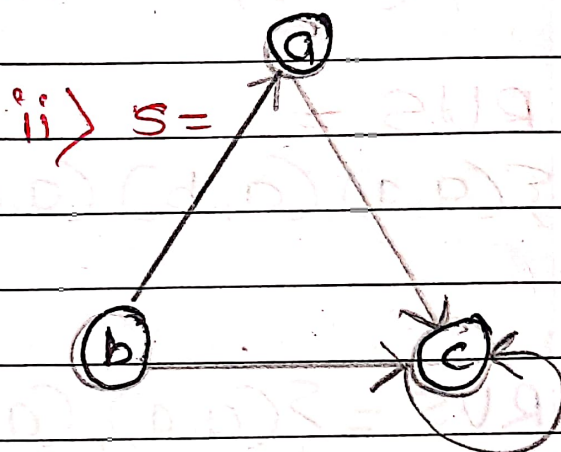
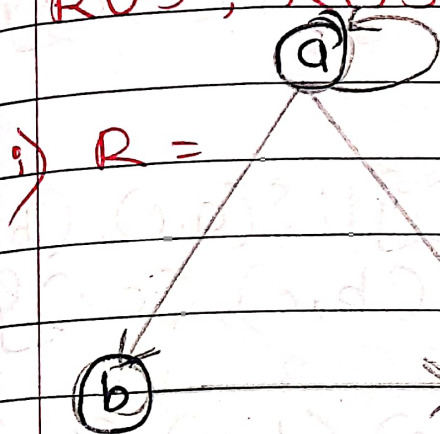
$$M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{(R \cup S)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M_{(R \cap S)} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{S^c} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

4) The digraphs of two relations R and S on the set $A = \{a, b, c\}$ are given below. Draw the digraphs of $\bar{R}$, $R \cup S$, $R \cap S$ and $R \circ S$.



Solⁿ Given $A = \{a, b, c\}$

By examining the given digraphs we note that

$$R = \{(a, a), (a, b), (a, c), (c, c)\}$$

and

$$S = \{(b, a), (b, c), (a, c), (c, c)\}$$

$\bar{R}$

$$\therefore \bar{R} = A \times A - R$$

$$A \times A = \{a, b, c\} \times \{a, b, c\}$$

$$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

$$\bar{R} = A \times A - R$$

$$\bar{R} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\} - \{(a, a), (a, b), (a, c), (b, c)\}$$

$$\bar{R} = \{(b, a), (b, b), (c, a), (c, b), (c, c)\}$$

$$R \cup S =$$

$$\Rightarrow \{(a, a), (a, b), (a, c), (b, c)\} \cup \{(a, c), (b, a), (b, c), (c, c)\}$$

$$R \cup S = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$$

$$R \cap S =$$

$$\Rightarrow \{(a, a), (a, b), (a, c), (b, c)\} \cap \{(a, c), (b, a), (b, c), (c, c)\}$$

$$R \cap S = \{(a, c), (b, c)\}$$

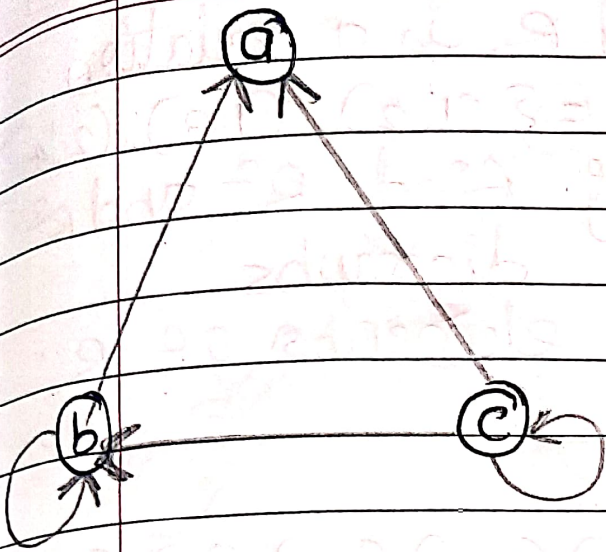
$$R^c =$$

$$R = \{(a, a), (a, b), (a, c), (b, c)\}$$

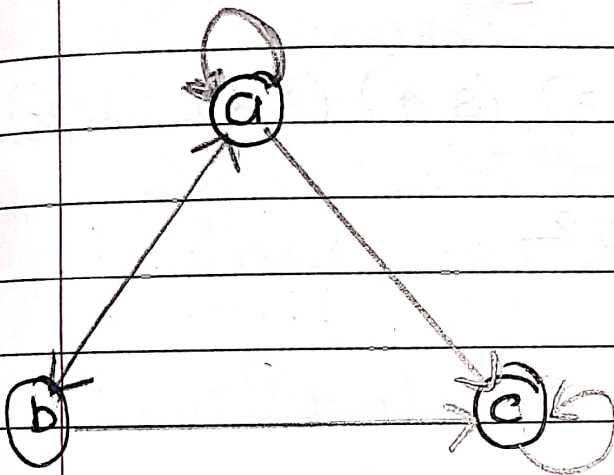
$$R^c = \{(a, a), (b, a), (c, a), (c, b)\}$$

The digraphs of these are as shown below.

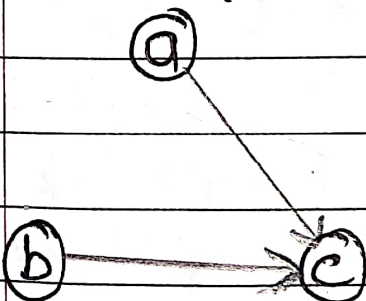
$$\overline{R} = \{(b, a), (b, b), (c, a), (c, b), (c, c)\}$$



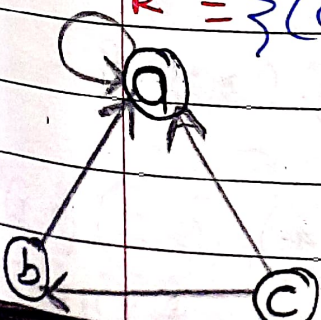
$$R \cup S = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$$



$$R \cap S = \{(a, c), (b, c)\}$$



$$R^c = \{(a, a), (b, a), (c, a), (c, b)\}$$



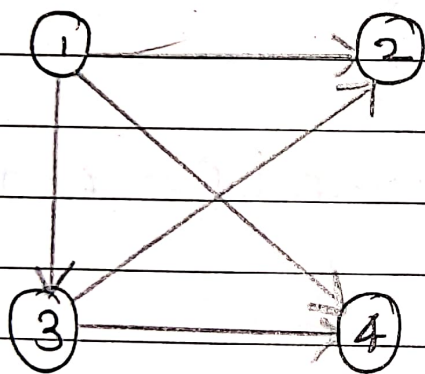
5) IF $A = \{1, 2, 3\}$ and R is a relation on A defined by $R = \{(1, 2), (1, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$ find R^2 and R^3 write down their digraphs

Solⁿ By examining the elements of R we find that

$$R^2 = R \circ R = \{(1, 2), (1, 3), (2, 4), (3, 2), (3, 3), (3, 4)\} \cup \{(1, 2), (1, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

$$R^2 = \{(1, 4), (1, 2), (1, 3), (3, 4), (3, 3), (3, 2)\}$$

The digraph of R^2



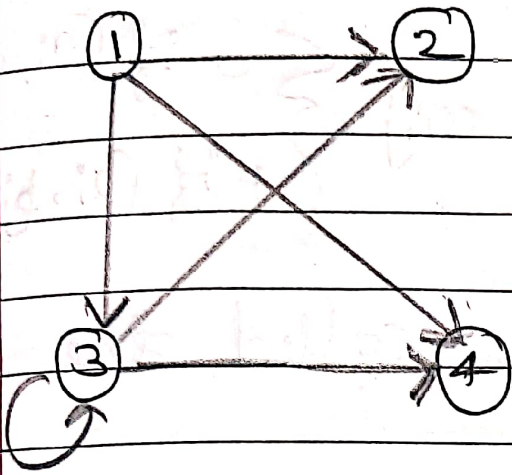
$$R^3 = R \circ R^2$$

$$R = \{(1, 2), (1, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

$$R^2 = \{(1, 4), (1, 2), (1, 3), (3, 4), (3, 3), (3, 2)\}$$

$$R^3 = R^2 \circ R = \{(1, 4), (1, 3), (3, 4), (3, 3), (3, 2), (1, 2)\}$$

The digraph of R^3



Representation of Relations:

Representation of Relation It has mainly two ways.

- 1) Relational Matrix
- 2) Digraph OR Directed graph

① Relational matrix OR Matrix representation of Relation:

Let $A = \{a_1, a_2, a_3, \dots, a_m\}$ and $B = \{b_1, b_2, b_3, \dots, b_n\}$ are finite sets containing m and n elements respectively and let R be a relation from A to B . Then R can be

represented by the $m \times n$ matrix.

$$M_R = [m_{ij}] \text{ where } m_{ij} = \begin{cases} 0 & \text{if } (a_i, b_j) \notin R \\ 1 & \text{if } (a_i, b_j) \in R \end{cases}$$

The matrix M_R is called the matrix of R .

Ex: 1) Let $A = \{1, 3, 4\}$ $R = \{(1,1), (1,3), (3,3), (4,4)\}$
 Then find matrix M_R ?

$$M_R = \begin{matrix} & \begin{matrix} 1 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

2) Represent the following Relation on set $\{1, 2, 3\}$ with a matrix.

$$R = \{(1,2), (2,1), (2,3), (3,3)\}$$

Sol: 2: Given $A = \{1, 2, 3\}$

$$R = \{(1,2), (2,1), (2,3), (3,3)\}$$

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

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Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

compute i) $A \cup B$

ii) $A \cap B$

Soln.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then Relation R

$$A = \{ (1, 1), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3) \}$$

$$B = \{ (1, 2), (1, 3), (2, 1), (2, 3), (3, 2) \}$$

$$i) A \cup B = \{ (1, 1), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3), (2, 1), (1, 2) \}$$

$$ii) A \cap B = \{ (1, 3), (2, 3), (3, 2) \}$$

④ Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by $(a, b) \in R$ if and only if $a \leq b$. Write down R as a set of ordered pairs. Also write down the matrix of this relation.

80/12 Given $A = \{1, 2, 3, 4\}$, $(a, b) \in R$ if and only if $a \leq b$ form the relation.

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$$

Then the matrix is

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

② Digraph OR Directed graph

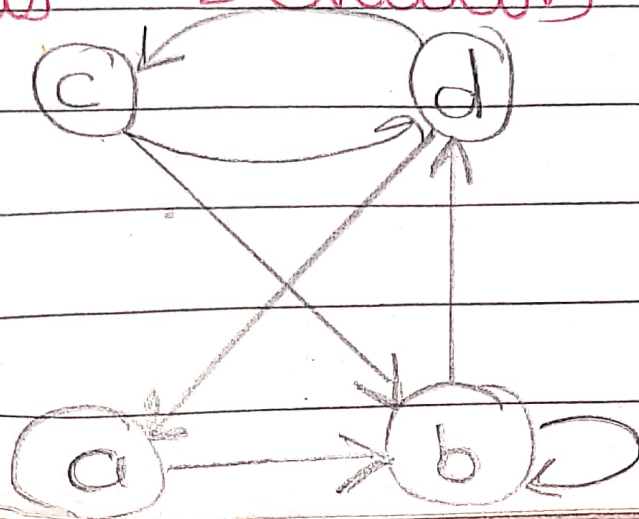
Draw a small circle for each element of A and label the circle with the corresponding element of A . These circles are called vertices or nodes. Draw an arrow called an edge from a vertex x to a vertex

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y if and only if $(x, y) \in R$. The resulting pictorial representation of R is called a directed graph or digraph of R .

Ex 10 Consider the set $A = \{a, b, c, d\}$ and the relation $R = \{(a, b), (b, b), (b, d), (c, d), (c, b), (d, a), (d, c)\}$ defined on A . The digraph of this relation is as shown below.



*
 ② Let $A = \{1, 2, 3, 4\}$ and Let R be the relation on A defined by xRy if and only if $y = 2x$

- Write down R as a set of ordered pairs
- Draw the digraph of R
- Determine the in-degree and out-degrees of the vertices in the digraph.

soln a) Write down R as a set of ordered pairs
 we observe that for $x, y \in A$ $(x, y) \in R$ if and only if $y = 2x$

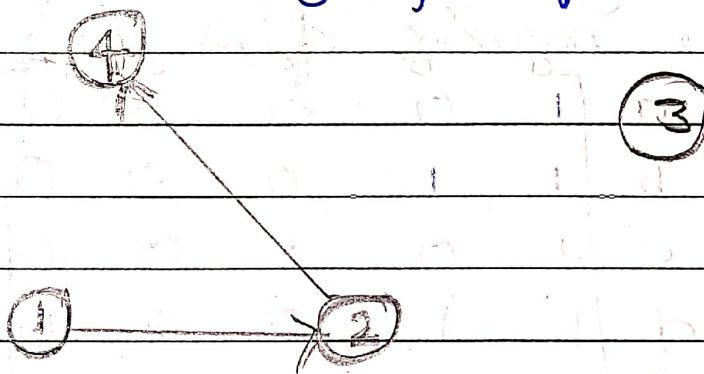
Thus $R = \{(1, 2), (2, 4)\}$ $y = 2x$

mean $y = 2x$

$y = 2 \times 1 = 2$, $y = 2 \times 2 = 4$. $y = 6$
 (3, 6)

b) Draw the digraph of R

soln



c) Determine the in-degree and out-degrees of the vertices in the digraph.

soln From the above digraph we note that 3 is an isolated vertex and that for the vertices 1, 2, 4 the in-degree and the out-degree are as shown in below.

vertex	1	2	4
In degree	0	1	1
Out degree	1	1	0

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Let $A = \{a, b, c, d, e\}$ and R be the relation on A defined by $R = \{(a, a), (b, a), (b, b), (c, a), (c, c), (d, a), (d, b), (d, d), (e, a), (e, b), (e, c), (e, e)\}$.
Represent the relation R as matrix and draw its digraph.

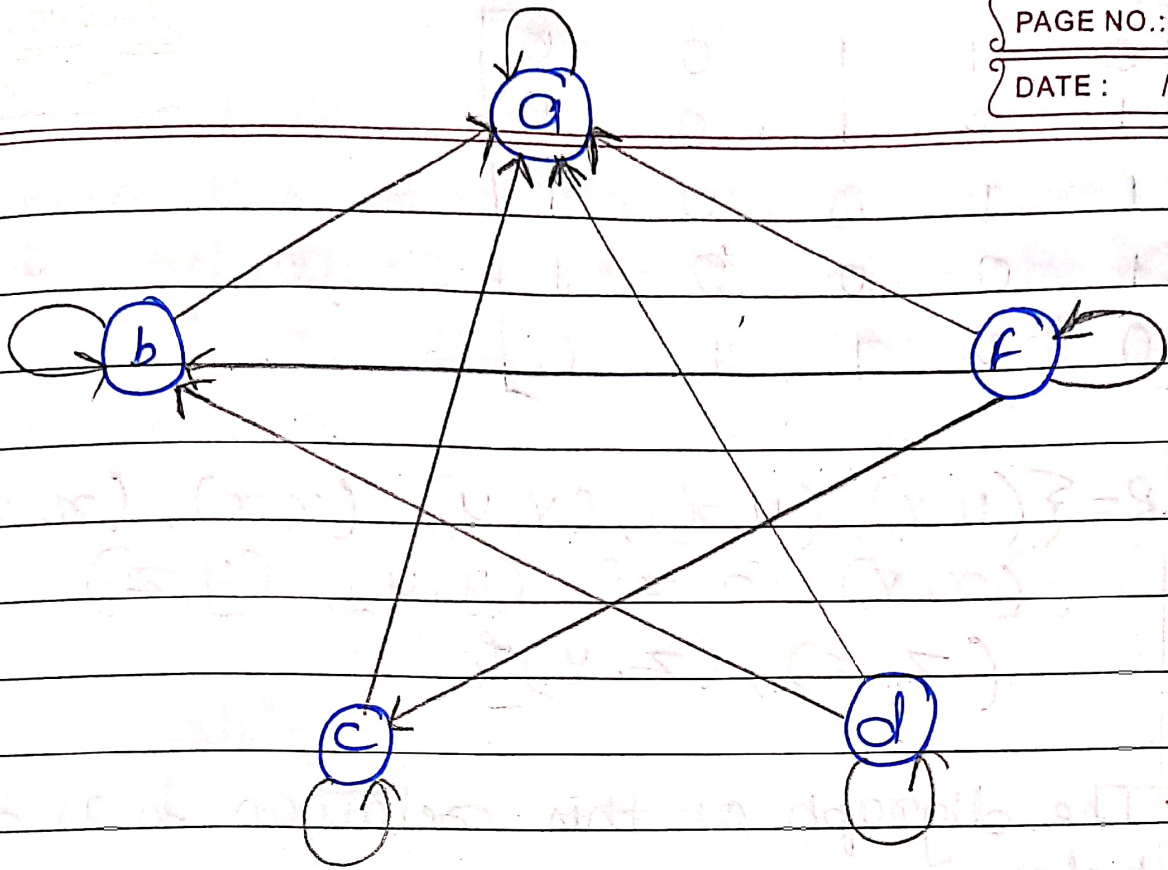
Soln: Given $A = \{a, b, c, d, e\}$ and R be the relation on A .

$R = \{(a, a), (b, a), (b, b), (c, a), (c, c), (d, a), (d, b), (d, d), (e, a), (e, b), (e, c), (e, e)\}$

By examining the elements of R we find that the matrix of R is.

$$M_R = \begin{matrix} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

The digraph of R is as shown below



4) Let $A = \{u, v, x, y, z\}$ and R be a relation on A whose matrix is as given below. Determine R and also draw the associated digraph.

$M(R) =$

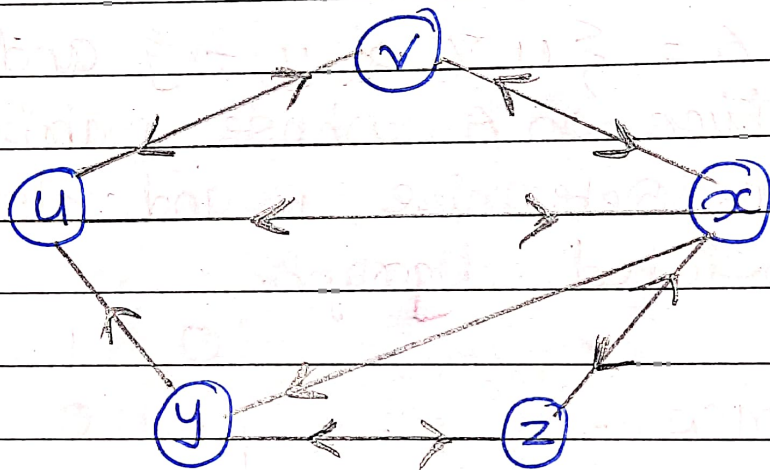
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Solⁿ By examining the elements of the given $M(R)$ we find that

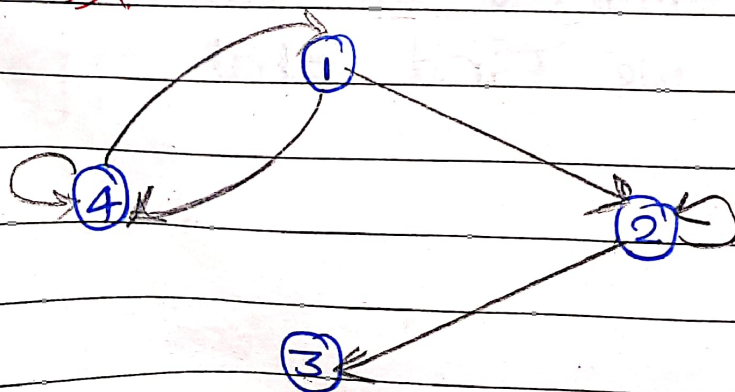
$$MR = \begin{matrix} & \begin{matrix} u & v & x & y & z \end{matrix} \\ \begin{matrix} u \\ v \\ x \\ y \\ z \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$R = \{(u, v), (u, x), (v, u), (v, x), (x, u), (x, v), (x, z), (y, u), (y, z), (z, x), (z, y)\}$$

The digraph of this relation is as shown below.



5) Find the relation represented by the digraph given below. Also write down its matrix.



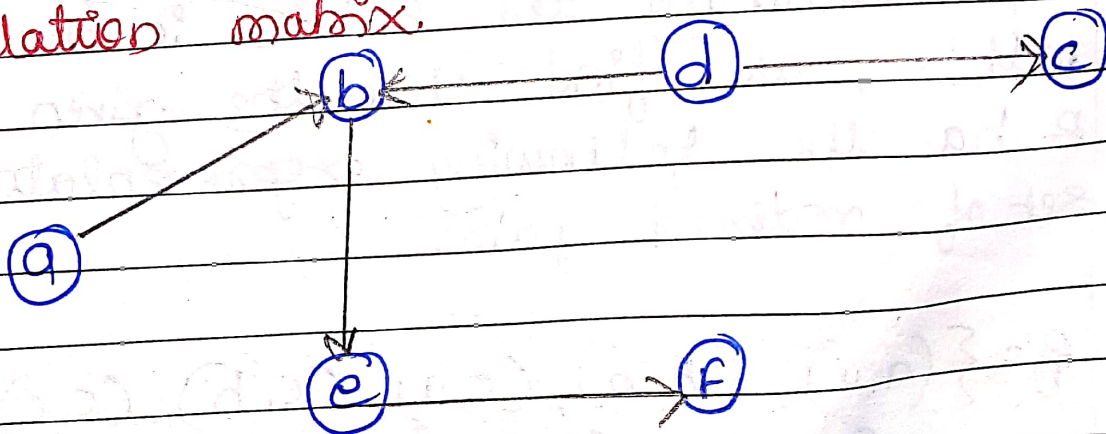
solⁿ By examine the given digraph which has four vertices we note that the relation R represented by it is defined on the set $A = \{1, 2, 3, 4\}$ and is given by

$$R = \{(1, 2), (1, 4), (2, 2), (2, 3), (4, 1), (4, 4)\}$$

The matrix of R is

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

6) For $A = \{a, b, c, d, e, f\}$ the digraph in figure represents a relation R on A . Determine R as well as its associated relation matrix.



solⁿ By examining the given digraph we find that

$$R = \{(a, b), (b, e), (e, f), (d, c), (d, b)\}$$

Also the matrix of R is given by

$$M(R) = \begin{matrix} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

1) Let $A = \{a, b, c, d\}$ and R be a relation on A that has the matrix,

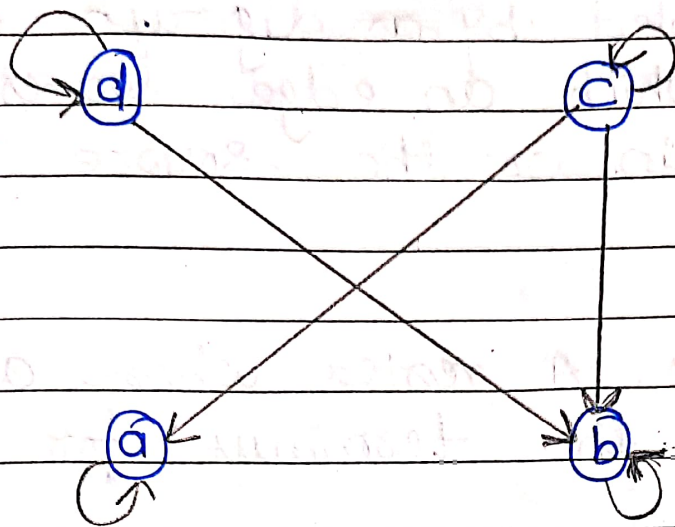
$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

construct the digraph of R and list the in-degrees and out-degrees of all vertices

Sol 2: By examining the entries in the given matrix, we find that the given relation R has the following representation as a set of ordered pairs.

$$R = \{(a, a), (b, b), (c, a), (c, b), (c, c), (d, b), (d, d)\}$$

The digraph of this relation is as shown below



The in degrees and out-degrees of the vertices are shown in the following Table

vertex	a	b	c	d
Indegree	2	3	1	1
out degree	1	1	3	2

8) Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by xRy if and only if " x divides y " written $x|y$.

- Write down R as a set of ordered pairs.
- Draw the digraph of R .
- Determine the in-degrees and out-degrees of the vertices in the digraph.

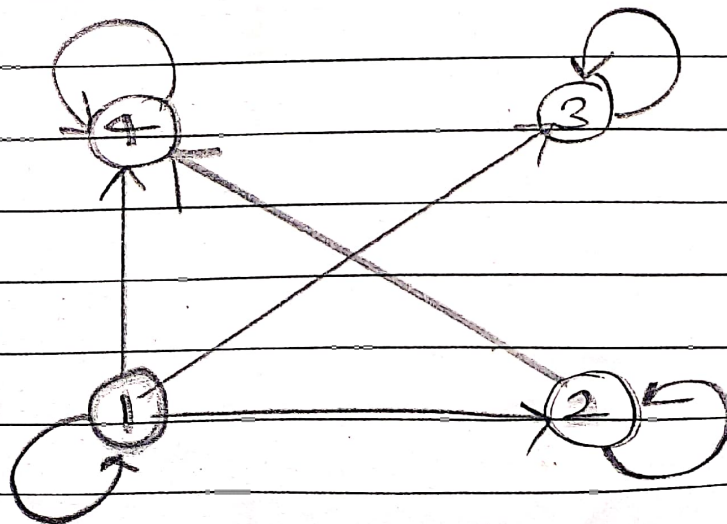
Solⁿ a) We observe that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

Hence.

$$\frac{x}{y} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{2}, \frac{2}{4}, \frac{3}{3}$$

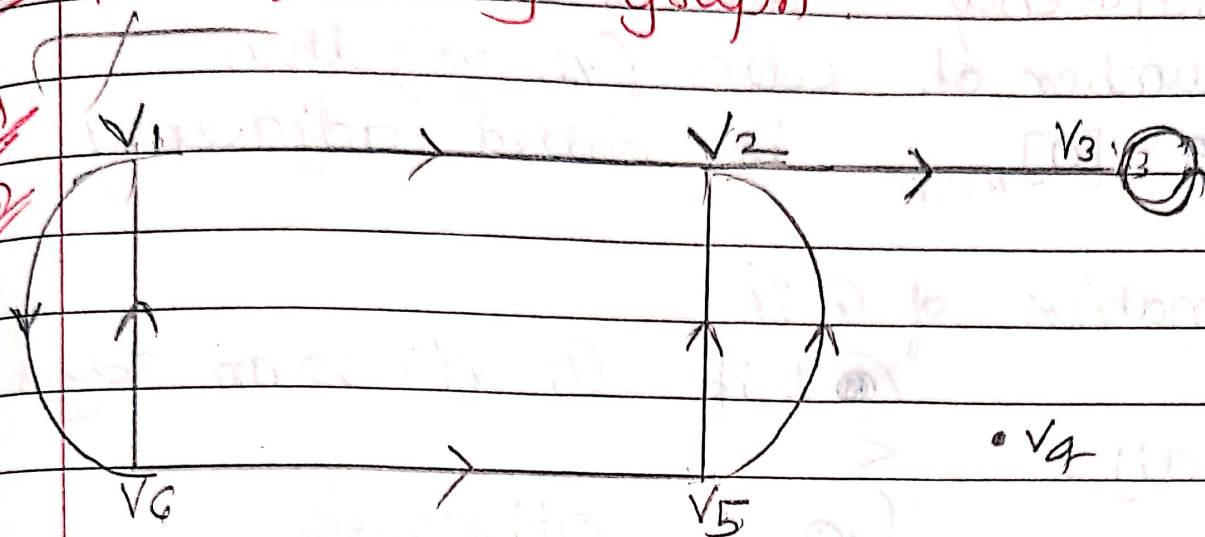
b) The digraph of R is as shown in below.



c) By examining the digraph we note that for the vertices 1, 2, 3, 4 the in-degree and the out-degrees are as shown in the following table.

vertex	In degree	out degree
1	1	4
2	2	2
3	2	1
4	3	1

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List out the indegree and outdegree for the following graph.



soln

vertex	v_1	v_2	v_3	v_4	v_5
Indegree	1	3	1	-	1
Outdegree	2	1	1	-	2

10) $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$. Represent the relation R as matrix and draw its digraph.

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Given $\{1, 2, 3, 4\}$

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$$

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$