

Unit II

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Counting & Advanced

Counting Technique

Basically there are two counting principles

- 1) Product Rule
- 2) Sum Rule.

1) **Product Rule**: The rule of product state that if one task can be performed in m ways the second task can be performed in n ways then the two task together can be performed in mn ways.

Q.) Suppose there are 7 different optional courses in Computer Science and 3 different optional courses in mathematics. How many options are there for a student who wants to take one optional course of each subject.

Soln. Number of optional courses in CS = 7
Number of optional courses in mathematics = 3

* Total Number of different options by taking one optional from of each subject

$$7(3) = ?$$

$$7 \times 3 = 21$$

2) $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$ Find $|A \times B|$

soln $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

$$|A \times B| = |A| \cdot |B|$$

$$= 4 \times 3$$

$$|A \times B| = 12$$

Note: Product rule in terms of sets
If $A_1, A_2, A_3, \dots, A_n$ are finite sets then the number of elements in the Cartesian product.
 $|A_1 \times A_2 \times A_3 \times \dots \times A_n| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_n|$

Imp
2m
2018
Sum Rule: - If one task can be performed in m ways and the second task can be performed in n ways and the two tasks can not be performed simultaneously then performing either task can be accomplished in any one of $m+n$ ways.

ex: 1) There are 5 BCA Colleges at Belgaum and 4 BCA Colleges at Hubli. Find the number of ways that a student

can take the admission to the college.

soln A student can be take admission to the BCA course at Belgaum in $m = 5$ ways. He can take admission to the course at Hubli in $n = 4$ ways.

Since he can not be take admission to the two colleges simultaneously then According to the sum rule

$$m+n = 5+4$$

$$m+n = 9 \text{ ways}$$

2) Suppose $A = \{1, 2, 3, 4\}$ $B = \{x, y, z\}$
find $|A \cup B|$

80/12 Given $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$
 $|A| = 4$ $|B| = 3$

$$\begin{aligned} \therefore |A \cup B| &= |A| + |B| \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

Note: Sum Rule in terms of Sets.

If $A_1, A_2, \dots, A_n$ are finite disjoint sets then the number of elements in the union of these sets.

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + |A_3| + \dots + |A_n|$$

Q A library contains 10 text books on mathematics and 20 text books on statistics. Find the number of way that a student can select a book on mathematics or statistics.

so/2 A student can select a book on mathematics in $m=10$ ways. He can select a book on statistics in $n=20$ ways.

According to the Sum Rule. He can select a book on mathematics or statistics in

$$\begin{aligned} m+n &= 10+20 \\ &= 30 \text{ ways.} \end{aligned}$$

Pigeonhole Principle:

statement: If n pigeons are assigned to m pigeonhole and $m < n$, then at least one pigeonhole contains two or more pigeons.

Proof: Suppose each pigeonhole contains at most 1 pigeon. Then at most m pigeons have been assigned. But since $m < n$ not all pigeons have been assigned pigeonholes. This is a Contradiction. At least one pigeonhole contains two or more pigeons.

Permutation And Combination.

Defn of Permutation: A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

Type 1: Permutation when all given objects are different.

Note: The number of permutation of n objects taken r at a time, when repetition of the object is not allowed then we use

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{where } r < n$$

2) The number of permutations of 'n' objects taken 'r' at a time when repetition of the object is allowed then we use

$${}_nP_r = n^r$$

Ex: 1) Find the number of words of 2 letters formed from 3 letters

Soln $n = 3, r = 2$

Then ${}_3P_2 = \frac{3!}{(3-2)!} = \frac{3 \times 2 \times 1}{1!} = 6$

2) How many different ways the letters of the word BOARD can be arranged? How many of these words begin with R? How many words can be formed using the letters of the word BOARD taken 3 at a time.

Soln The word BOARD consist of 5 different letters B, O, A, R, D. These 5 different letters can be arranged among themselves in ${}_5P_5$

$${}_5P_5 = \frac{5!}{(5-5)!}$$

$$= \frac{5!}{0!}$$

$$10 \text{ OR } 0! = 1$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{1} = 120 \text{ ways}$$

If the word begin with R then the remaining four letter can be arranged among themselves in $4P_4$

$$4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24 \text{ ways}$$

From given 5 different letters any three different letters can be arranged in $5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$

$$\frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60 \text{ ways}$$

3) How many 2-digit numbers can be formed with the digit 3, 5, 6, 7, 8 when the digit in a number may be repeated?

$$\text{Soln } nP_r = n^r = 5^2, \quad r=2, \quad n=5$$

then

$$= 5^2 = 25 \text{ ways}$$

Type 2:

Permutation when all objects are not distinct.

Note: The number of permutation of n objects where p objects are of one kind, q are of other kind then permutation is

$$\frac{n!}{p! q!}$$

ex: 1) Find the number of arrangement from all the letters of the word. MASSASAUGA.

sol: Number of letter A i.e. $p = 4$
Number of letter S i.e. $q = 3$

Total number of letters $n = 10$

The number of arrangement or permutation is

$$\frac{n!}{p! q!} = \frac{10!}{4! \times 3!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 5}{1} = 25200.$$

Find the number of permutation of the word PEPPER and

1) All the letters of the word PEPPER
2) With all p's together of the word PEPPER

Solⁿ The number of words in given word are 6.

Number of permutation

$$P \rightarrow 3$$

$$E \rightarrow 2$$

then

$$\frac{n!}{p! q!} = \frac{6!}{3! 2!}$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 6 \times 5 \times 2 = 60$$

2) P P P.

Here 3 P's now becomes a single element and the number of letters now count to 4

$$E \rightarrow 2 \quad \times \quad \frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$= 12$$

3) Find the number of permutations of the word KARNATAKA and permutations if all A's together

Solⁿ The number of words in given word are 9.

No of permutations

$$A \rightarrow 4 = p$$

$$K \rightarrow 2 = q$$

$$\text{then } \frac{n!}{p! q!}$$

$$\frac{9!}{4! 2!}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1}$$

$$= \frac{9 \times 8 \times 7 \times 3 \times 5}{1} = 7,560$$

And the permutation if all A's together.

Here A A A A

Here 4 A's now becomes a single element and the number of letters now count to 6

$$K = 2 \quad \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 360$$

4) Find the number of permutation of the letters of the word MASSASAUGA in how many of these all four A's are together? And how many of them begin with S?

Solⁿ - The number of letters in the given word are 10

4 A's together.

AAAA. - - - -

Here 4 A's now become a single element and the number of letters now count to 7

$$S = 3$$

$$\frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$$

$$= 7 \times 6 \times 5 \times 4 = 840$$

MASSASAUGA

The word which begin with S?

SSS. - - - -

The 3 S's now become a single element and the number of letters now count to 8!

(3 S's + 7 others)

Also when word begins with 3 S's the first place becomes fixed and we have to arrange remaining 7 letters

In these 7 letters A occurs
four

$${}^7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!}$$

$$\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 840$$

$$7 \times 6 \times 5 \times 4 = 840$$

2019
5m 5) Find the number of permutation of
the letter word "ENGINEERING"
In how many arrangements the 3 E's
are together?

soln The total number of letters are 11
Number of permutation
E occurs thrice, N occurs thrice
I occurs twice.

$$E = 3 \quad N = 3 \quad I = 2$$

$$n = 11$$

Then Number of permutation

$$\frac{n!}{p!q!r!} = \frac{11!}{3!3!2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 2 \times 1}$$

$$\frac{11 \times 10 \times 9 \times 8 \times 7 \times 2 \times 5}{1} = 554,400$$

3 E's are together
EEE - - - -

Here 3 E's now becomes a single

element and the number of letters now count to 9

$$N = 3, I = 2, G = 2$$

$$= \frac{9!}{3! 2! 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4} \times \cancel{3}!}{\cancel{3}! \times \cancel{2}! \times \cancel{2}! \times 1 \times 1 \times 1}$$

$$= 9 \times 8 \times 7 \times 6 \times 5$$

$$= 15,120$$

Combination:

A selection of number obj-
object taking some of or all of
them at a time is called
combination

Type 1:

Total number of combination or
selection of n different objects
taken r at a time

$${}^n C_r = \frac{n!}{(n-r)! r!}$$

Note: ${}^n C_n = 1$

* ${}^n C_0 = 1$

1) Find the value of:

2) ${}^7 C_3 =$

$$= {}^7 C_3 = \frac{7!}{(7-3)! 3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4! 3!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = \frac{7 \times 5}{1} = 35$$

$$2) \quad {}^{10}C_7 = \frac{10!}{(10-7)! \cdot 7!} = \frac{10!}{3! \cdot 7!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{5 \times 3 \times 8}{1} = 120$$

Imp 2021 3) In how many ways we can distribute 10 different marbles among 6 distinct containers?

80/12 Number of ways of distributing 10 different marbles among 6 distinct containers is

$${}^{10}C_6 = \frac{10!}{(10-6)! \cdot 6!} = \frac{10!}{4! \cdot 6!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{10 \times 3 \times 7}{1}$$

$$= 210 \text{ ways}$$

2018 500 Imp 4) Out of 6 boys and 4 girls a committee of 6 is to be formed. In how many ways can this be done if the committee contains

i) 2 girls

ii) At least 2 girls

Sol 2

We have

Number of girls = 4

Number of boys = 6

∴ 2 girls

Since Committee has to consist

of 2 girls

∴ The Committee consist of 2 girls and 4 boys.

2 girls out of 4 can be selected in 4C_2 ways.4 boys out of 6 can be selected in 6C_4 ∴ Required number of ways
 ${}^4C_2 \times {}^6C_4$

$$= \frac{4!}{(4-2)! 2!} \times \frac{6!}{(6-4)! 4!}$$

$$= \frac{4!}{2! 2!} \times \frac{6!}{2! 4!}$$

$$= \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} \times \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{2 \times 3}{1} \times \frac{3 \times 5}{1}$$

$$= 6 \times 15$$

$$= 90 \text{ ways}$$

ii) at least 2 girls

Since the committee has to consist of at least 2 girls the committee can consist of

- a) 2 girls and 4 boys
4 girls and 2 boys

a) 2 girls and 4 boys can be selected in ${}^4C_2 \times {}^6C_4$ ways

b) 4 girls and 2 boys can be selected in ${}^4C_4 \times {}^6C_2$ ways

$\therefore$ Required number of ways

$$\left({}^4C_2 \times {}^6C_4 \right) + \left({}^4C_4 \times {}^6C_2 \right)$$

$$= \left(\frac{4!}{(4-2)! 2!} \times \frac{6!}{(6-4)! 4!} \right) + \left(\frac{4!}{(4-4)! 4!} \times \frac{6!}{(6-2)! 2!} \right)$$

$$= \left(\frac{4!}{2! 2!} \times \frac{6!}{2! 4!} \right) + \left(\frac{4!}{0! 4!} \times \frac{6!}{4! 2!} \right)$$

$$= \left[\frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} \times \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 4 \times 3 \times 2 \times 1} \right] + \left[\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 2 \times 1} \right]$$

$$= \left[\frac{2 \times 3}{1} \times \frac{3 \times 5}{1} \right] + \left[\frac{3 \times 5}{1} \right]$$

$$= [6 \times 15] + [15]$$

$$= 90 + 15$$

$$= 105 \text{ ways}$$

5) A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of

i) Exactly 3 girls

ii) At least 3 girls

iii) At most 3 girls

Solⁿ Number of girls = 4
Number of boys = 9

i) Exactly 3 girls

Since committee has to consist exactly 3 girls

∴ The committee can consist of 3 girls and 4 boys.

3 girls out of 4 can be selected in 4C_3 ways.

4 boys out of 9 can be selected in 9C_4 ways.

∴ Required number of ways.

$${}^4C_3 \times {}^9C_4$$

$$= \frac{4!}{(4-3)! 3!} \times \frac{9!}{(9-4)! 4!}$$

$$= \frac{4!}{1! 3!} \times \frac{9!}{5! 4!}$$

$$= \frac{4 \times 3 \times 2 \times 1}{1 \times 3 \times 2 \times 1} \times \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1}$$

$$= 4 \times 9 \times 2 \times 7$$

$$= 504 \text{ ways}$$

ii) Atleast 3 girls:

Since the committee has to consist of atleast 3 girls the committee can consist of.

a) 3 girls and 4 boys.

b) 4 girls and 3 boys

a) 3 girls & 4 boys can be selected in
 ${}^4C_3 \times {}^9C_4$

b) 4 girls & 3 boys can be selected in
 ${}^4C_4 \times {}^9C_3$ ways.

∴ Required no. of ways.

$$({}^4C_3 \times {}^9C_4) + ({}^4C_4 \times {}^9C_3)$$

$$\left(\frac{4!}{(4-3)! \cdot 3!} \times \frac{9!}{(9-4)! \cdot 4!} \right) + \left(\frac{4!}{(4-4)! \cdot 4!} \times \frac{9!}{(9-3)! \cdot 3!} \right)$$

$$= \left(\frac{4!}{1! \cdot 3!} \times \frac{9!}{5! \cdot 4!} \right) + \frac{4!}{0! \cdot 4!} \times \frac{9!}{6! \cdot 3!}$$

$$= \left(\frac{4 \times 3 \times 2 \times 1}{1 \times 3 \times 2 \times 1} \times \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} \right) + \left(\frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} \right)$$

$$= (4 \times 9 \times 2 \times 7) + (3 \times 4 \times 7)$$

$$= 588 \text{ ways.}$$

(iii) Atmost 3 girls

In this cases Committee consist of 3 girls, 2 girls, 1 girl, or no girl.

Number of girls	Number of boys	Number of ways
3	4	${}^4C_3 \times {}^9C_4$
2	5	${}^4C_2 \times {}^9C_5$
1	6	${}^4C_1 \times {}^9C_6$
0	7	${}^4C_0 \times {}^9C_7$

∴ Required No. of ways.

$$= ({}^4C_3 \times {}^9C_4) + ({}^4C_2 \times {}^9C_5) + ({}^4C_1 \times {}^9C_6) + ({}^4C_0 \times {}^9C_7)$$

$$= \left[\frac{4!}{(4-3)! 3!} \times \frac{9!}{(9-4)! 4!} \right] + \left[\frac{4!}{(4-2)! 2!} \times \frac{9!}{(9-5)! 5!} \right]$$

$$+ \left[\frac{4!}{(4-1)! 1!} \times \frac{9!}{(9-6)! 6!} \right] + \left[\frac{4!}{(4-0)! 0!} \times \frac{9!}{(9-7)! 7!} \right]$$

$$= \left[\frac{4!}{1! 3!} \times \frac{9!}{5! 4!} \right] + \left[\frac{4!}{2! 2!} \times \frac{9!}{4! 5!} \right] +$$

$$\left[\frac{4!}{3! 1!} \times \frac{9!}{3! 6!} \right] + \left[\frac{4!}{4! 0!} \times \frac{9!}{2! 7!} \right]$$

$$\begin{aligned}
 &= \left[\frac{4 \times 3!}{1 \times 3!} \times \frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1} \right] + \left[\frac{4 \times 3 \times 2!}{2! \times 2 \times 1} \times \frac{9 \times 8 \times 7 \times 6!}{5! \times 4 \times 3 \times 2 \times 1} \right] \\
 &\quad + \left[\frac{4 \times 3!}{3! \times 1} \times \frac{9 \times 8 \times 7 \times 6!}{6! \times 3 \times 2 \times 1} \right] + \left[\frac{9 \times 8 \times 7!}{7! \times 2 \times 1} \right] \\
 &= (4 \times 126) + (6 \times 126) + (4 \times 3 \times 4 \times 7) + (9 \times 4) \\
 &= 504 + 756 + 336 + 36 \\
 &= 1,632 \text{ ways.}
 \end{aligned}$$

Type 2 : The combination of 'n' object taking 'r' at a time with repetition allowed is given by,

$${}^{n+r-1}C_r = \frac{(n+r-1)!}{(n-1)! r!}$$

above formula is also used to find "no. of ways we can distribute 'r' similar things in 'n' numbered boxes.

1) Example :

In how many ways one can distribute 12 identical pencils to 5 children so that every child gets at least one pencil

80/2 Here $n = 12$, $r = 5$

The pencils can be distributed to the children is

$${}_{n+r-1}C_r = \frac{(n+r-1)!}{(n-1)! r!}$$

$${}_{12+5-1}C_5 = \frac{(12+5-1)!}{(12-1)! 5!} = \frac{17!}{11! 5!}$$

$$= \frac{16 \times 15 \times 14 \times 13 \times 12 \times 11!}{11! \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 2 \times 14 \times 13 \times 12$$

$$= 4,368 \text{ ways.}$$

2) In how many ways we can distribute 10 identical marbles among 6 distinct containers

80/2 $n = 10$, $r = 6$

$${}_{10+6-1}C_6 = \frac{(10+6-1)!}{(10-1)! 6!}$$

$$\frac{(16-1)!}{9! 6!} = \frac{15!}{9! 6!}$$

$$= \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9!}{9! \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 7 \times 13 \times 11 \times 5$$

$$= 5005 \text{ ways}$$

Important Combinations Formula:

i) Number of combinations of 'n' different things taken 'r' at a time when 'p' particular things are always included

$$= \frac{n-p}{r-p} C$$

ii) Number of combination of n different things taken r at time when p particular things are always to be excluded.

$$= \frac{n-p}{r} C$$

Q) In how many ways can a cricket eleven be selected out of 15 players? In how many of them will a particular player be

i) Included ii) excluded.

80/2 It is required to select 11 players out of 15
This can be done in.

$${}^{15}C_{11} = \frac{15!}{(15-11)!11!} = \frac{15!}{4!11!}$$

$$= \frac{15 \times 14 \times 13 \times 12 \times 11!}{4 \times 3 \times 2 \times 1 \times 11!} = 15 \times 7 \times 13 = 1365 \text{ ways.}$$

i) since a particular player is to be included

80/2 We have $n = 15$, $r = 11$, $p = 1$

$${}^{n-p}C_{r-p} = {}^{15-1}C_{11-1} = {}^{14}C_{10}$$

$$= \frac{14!}{(14-10)!10!} = \frac{14!}{4!10!}$$

$$= \frac{14 \times 13 \times 12 \times 11 \times 10!}{4 \times 3 \times 2 \times 1 \times 10!}$$

$$= 7 \times 13 \times 11$$

$$= 1001 \text{ ways}$$

ii) since a particular is to be excluded:

$$n - p = \frac{15-1}{C_{11}} = \frac{14}{C_{11}}$$

$$= \frac{14!}{(14-11)! 11!} = \frac{14!}{3! 11!}$$

$$= \frac{14 \times 13 \times 12 \times 11!}{3 \times 2 \times 1 \times 11!}$$

$$= 14 \times 13 \times 2$$

$$= 364 \text{ ways}$$

Binomial Theorem:

An algebraic expression which contains two terms separated by plus or minus sign is called a binomial.

ex: $x+a$, $\frac{3x^2}{2} - \frac{1}{5y}$ are Binomials.

The expansion of the binomial to any power 'n' is called the binomial theorem

i.e.

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_r a^{n-r} b^r + \dots + {}^nC_n a^0 b^n$$

Note! The number of terms in expansion is $(n+1)$

2) The $(r+1)^{th}$ term in the binomial expansion is called the general term and is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

3) The co-efficient of $x^{n-r} y^r$ where $0 \leq r \leq n$ is the number of different ways in which we can select $(n-r)$

x 's and r no. of y 's from the n available factors. The total no. of selection is.

$${}^n C_{n-r} \quad \text{OR} \quad {}^n C_r \quad \text{This term is}$$

called Binomial Co-efficient.

4) Middle term: In $(a+b)^n$

a) When n is even the middle term is $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term.

b) When n is odd, the middle term is $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and $\left(\frac{n+3}{2}\right)^{\text{th}}$ term.

Example:

i) Expand using Binomial theorem $(a+b)^3$

$$(a+b)^3 = {}^3 C_0 a^3 b^0 + {}^3 C_1 a^2 b^1 + {}^3 C_2 a^1 b^2 + {}^3 C_3 a^0 b^3$$

$$= (1)a^3 b^0 + \frac{3!}{(3-1)!1!} a^2 b^1 + \frac{3!}{(3-2)!2!} a^1 b^2 + \frac{3!}{(3-3)!3!} a^0 b^3$$

$$= a^3 b^0 + \frac{3!}{2!1!} a^2 b^1 + \frac{3!}{1!2!} a^1 b^2 + \frac{3!}{0!3!} a^0 b^3$$

$$= a^3 b^0 + \frac{3 \times 2 \times 1}{2 \times 1 \times 1} a^2 b^1 + \frac{3 \times 2 \times 1}{1 \times 2 \times 1} a^1 b^2 + \frac{3 \times 2 \times 1}{0! \times 3 \times 2 \times 1} a^0 b^3$$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

2) Find the 4th term in expansion of

$$(x - 2y)^{12}$$

solⁿ We have

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$\text{Given } (x - 2y)^{12}$$

$$a = x \quad b = -2y \quad r+1 = 4, \quad n = 12$$

$$T_4 = {}^{12}C_3 x^{12-3} (-2y)^3$$

$$T_4 = \frac{12!}{(12-3)! 3!} x^9 (-8y^3)$$

$$= T_4 = \frac{12 \times 11 \times 10 \times 9!}{9! \times 3 \times 2 \times 1} x^9 (-8y^3)$$

$$= T_4 = (2 \times 11 \times 10) x^9 (-8y^3)$$

$$T_4 = -1760 x^9 y^3$$

3) Find the 7th term in expansion of

$$(3x^2 - y/3)^9$$

$$\text{solⁿ We have } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Here

$$(3x^2 - y/3)^9$$

$$n = 9, \quad a = 3x^2, \quad b = -y/3, \quad r+1 = 7$$

$$T_7 = {}^9C_6 (3x^2)^{9-6} (-y/3)^6$$

$$T_7 = {}^9C_6 (3x^2)^3 (-y/3)^6$$

$$T_7 = \frac{9!}{(9-6)! 6!} 27x^6 \frac{y^6}{3^3 \times 3^3}$$

$$T_7 = \frac{9 \times 8 \times 7 \times 6!}{3 \times 2 \times 1! 6!} 27x^6 \frac{y^6}{27 \times 27}$$

$$T_7 = \frac{3 \times 4 \times 7}{27 \times 27} x^6 y^6$$

$$T_7 = \frac{28}{9} x^6 y^6$$

4) Find the middle terms in the expansion of

1) $\left(3 - \frac{x^3}{6}\right)^7$

2) $\left(\frac{x}{3} + 9y\right)^{10}$

1) $\left(3 - \frac{x^3}{6}\right)^7$

solⁿ

$n = 7$ (odd)

Middle term = $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and

$\left(\frac{n+3}{2}\right)^{\text{th}}$ term.

Here $n = 7$

$\left(\frac{7+1}{2}\right)^{\text{th}}$ terms and $\left(\frac{7+3}{2}\right)^{\text{th}}$ term

$\left(\frac{8}{2}\right)^{\text{th}}$ terms & $\left(\frac{10}{2}\right)^{\text{th}}$ terms

$(4)^{\text{th}}$ term & 5^{th} term

then T_4 and T_5 .

Now general term.

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$n = 7, a = 3, b = -x^3/6$$

$$r+1 = 4 \quad n-r$$

$$r = 4-1 = 3$$

$$r = 3 \quad = 4$$

$$\therefore T_4 = {}^7C_3 (3)^4 \left(\frac{-x^3}{6}\right)^3$$

$$= \frac{-7!}{(7-3)! 3!} \times 3^4 \times \frac{x^9}{6^3}$$

$$= \frac{-7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} \times \frac{81}{216} \times x^9$$

$$= \frac{-7 \times 5}{216} \times x^9$$

$$= \frac{-35 \times 81}{216} x^9$$

$$= \frac{-105}{8} x^9$$

$$\text{Now } T_5 = {}^nC_r a^{n-r} b^r$$

$$r+1=5 \quad n=7, \quad a=3, \quad b=-\frac{x^3}{6}$$

$$r=5-1 \quad n-r=7-4$$

$$r=4 \quad n-r=3$$

$$= {}^7C_4 3^{7-4} \left(-\frac{x^3}{6} \right)^4$$

$$= \frac{7!}{(7-4)! 4!} 3^3 \frac{-x^{12}}{6^4}$$

$$= \frac{7!}{3! 4!} \times 27 \times \frac{+x^{12}}{1296}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} \times 27 \times \frac{+x^{12}}{1296}$$

$$= \frac{+7 \times 5 \times 27 \times x^{12}}{1296}$$

$$= \frac{+35}{48} x^{12}$$

$$ii) \left(3 - \frac{x^3}{6} \right)^{10}$$

$$\text{soln} \left(3 - \frac{x^3}{6} \right)^{10}$$

$$n = 10 \quad (\text{even})$$

middle term when n is even
then $\left(\frac{n}{2} + 1 \right) = \left(\frac{10}{2} + 1 \right) = \frac{10+2}{2} = \frac{12}{2} = 6$

$$n = 6$$

$\therefore T_6$ is the middle term.

Now General term.

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$r+1 = 6 \quad a = 3, \quad b = \frac{-x^3}{6}$$

$$r = 5 \quad n - r$$

$$10 - 5 = 5$$

$$T_6 = {}^{10}C_5 3^{10-5} \left(\frac{-x^3}{6} \right)^5$$

$$T_6 = \frac{10!}{(10-5)! 5!} \times 3^5 \frac{-x^{15}}{6^5}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! 5 \times 4 \times 3 \times 2 \times 1} \times 243 \frac{-x^{15}}{7776}$$

$$\frac{3 \times 2 \times 7 \times 6}{1} \times \frac{-x^{15}}{2592} = \frac{-x^{15} 7}{72}$$

$$\frac{1296}{432} = 3$$

5) Find the co-efficient of a^5b^2 in the expansion of $(2a-3b)^7$

soln We know that

$$T_{r+1} = {}^nC_r a^{n-r} b^r \rightarrow \textcircled{1}$$

$$n=7, a=2a, b=-3b$$

$$\text{in } (2a-3b)^7$$

we

$$T_{r+1} = {}^7C_r a^{7-r} b^r \rightarrow \textcircled{2}$$

from $\textcircled{1}$ & $\textcircled{2}$

Also given a^5b^2

then

$$a^{7-r} = a^5$$

$$7-r = 5$$

$$-r = 5-7$$

$$r = 2$$

Now from $\textcircled{1}$

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

$$T_{2+1} = {}^7C_2 (2a)^{7-2} (-3b)^2$$

$$= \frac{7!}{(7-2)! 2!} \times (2a)^5 (-3b)^2$$

$$= \frac{7 \times 6 \times 5!}{5! \times 2 \times 1} \times 2^5 a^5 (-3)^2 b^2$$

$$= 7 \times 3 \times 2^5 \times a^5 \times 9 \times b^2$$

$$= 6048 a^5 b^2$$

⑥ Find the co-efficient of $x^5 y^2$ in the expansion of $(x+y)^7$

soln We know that

$$T_{r+1} = {}^n C_r a^{n-r} b^r \rightarrow (1)$$

The expansion of $(x+y)^7$

$$n=7, a=x, b=y$$

$$\text{Also given } x^5 y^2 \rightarrow (2)$$

$\therefore$ from (1) and (2)

$${}^7 C_{r+1} = x^5$$

$$7-r = 5$$

$$-r = 5-7$$

$$-r = -2$$

$$r = 2$$

$$T_{2+1} = {}^7 C_2 x^{7-2} y^2$$

$$T_3 = \frac{7!}{(7-2)! 2!} x^5 y^2$$

$$= \frac{7 \times 6 \times 5}{5! \times 2 \times 1} \times x^5 y^2$$

$$= 7 \times 3 \times x^5 y^2$$

$$= 21 x^5 y^2$$

Generating permutation and Combination:

Permutation with repetition:

Generalization:

Number of ways - $\frac{n!}{n_1! \cdot n_2! \cdot n_3! \dots} = n_k!$

ex: POOTA

$$P=1, O=2, T=1, A=1$$

Total number of letters = 5

$$\text{Number of ways} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$= 5 \times 4 \times 3$$

$$= 60$$

Definition of permutation generation:

An arrangement of its members into a sequence or linear order or if the set is already ordered a rearrangement of its elements.

Generating Combination:

A combination $a_1, a_2, \dots, a_r$ in lexicographic order is given for a set $\{1, 2, 3, \dots, n\}$. In the set $\{1, 2, 3, 4, 5, 6\}$ a 4-combination could be $\{1, 2, 5, 6\}$ to obtain the next largest combination find the last a_i in the combination so that $a_i! = n - r + i$.

Introduction:

Recurrence Relation:

When the problem is to find an explicit formula for a recursively defined sequence the recursive formula is called a Recurrence Relation.

Application of Recurrence Relation:

- 1) Recurrence Relation are used to reduce complicated problems to an iterative process based on simpler versions of the problem.
- 2) A recurrence relation is an equation which represents a sequence based on some rule. It help in finding the subsequent term (next term) dependent upon the preceding term (previous term).

Defn of Recurrence Relation:

A formula which defines any terms of a sequence by its previous terms is called recursive formula and the relation is called a recurrence relation.

ex: Consider the sequence, 3, 8, 13, 18, 23, ...
then

$$a_1 = 3, a_2 = 8, a_3 = 13, a_4 = 18, a_5 = 23$$

$$a_1 = 3$$

$$a_1 = 3$$

$$a_2 = 8 = 3 + 5 = a_1 + 5$$

$$a_3 = 13 = 8 + 5 = a_2 + 5$$

$$a_4 = 18 = 13 + 5 = a_3 + 5$$

$$a_5 = 23 = 18 + 5 = a_4 + 5$$

!

$$a_n = a_{n-1} + 5$$

Hence $a_n = a_{n-1} + 5$, for $n \geq 2$ & $a_1 = 3$

← This formula is called recursive formula and relation is called recurrence relation or difference equation.

→ The big beginning information $a_1 = 3$ is called the initial condition or boundary condition

Note: very important note is that the recurrence relation can be written in the different form.

Imp 1) $a_{x+3} + 3a_{x+2} + 4a_{x+1} + 8a_x = 0$ or $f(x)$

2) $y_{h+3} + 3y_{h+2} + 4y_{h+1} + 8y_h = 0$ or $f(h)$

OR

Imp 3) $u_{x+3} + 3u_{x+2} + 4u_{x+1} + 8u_x = 0$ or $f(x)$

OR

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4) $F(x+3h) + 3F(x+2h) + 4F(x+h) +$

$\downarrow$ $8F(x) = 0 \underline{00} K(x)$

$a_x = F(x)$

$a_{x+1} = F(x+h)$

$a_{x+2} = F(x+2h)$

$a_{x+3} = F(x+3h)$

order of the recurrence relation :

The difference between highest and lowest subscript of $F(x)$ or a_x or y_h or u_x is called order of the recurrence relation.

ex: 1) $a_x + 3a_{x-1} = 0$

sol: Here $x =$ highest and $x-1 =$ lowest then

Then difference $x - (x-1) = \cancel{x} - \cancel{x} + 1 = 1$

The order is 1

2) y_{h+3}

2) $5F(x+2) + F(x+1) + 7F(x) = K(x)$

sol: Here $x+2 =$ highest and $x =$ lowest then difference is

$\cancel{x}+2 - \cancel{x} = 2$

The order is 2

$$3) y_{h+3} + 3y_{h+2} + 4y_{h+1} + 8y_h = 0$$

solⁿ Here $h+3 =$ highest and h is lowest
then difference is

$$(h+3) - h = h+3-h = 3$$

The order is 3

$$4) u_{x+4} + 5u_{x+1} = 0$$

solⁿ Here $x+4 =$ highest and $x+1 =$ lowest
Then difference is

$$\begin{aligned} & (x+4) - (x+1) \\ &= x+4-x-1 = 3 \end{aligned}$$

~~Then order is 3~~ Then order is 3

Degree of the recurrence relation:

The degree of a recurrence relation is defined by the highest power of a_x , or $f(n)$ or u_n or $f(n)$

ex^o:

$$1) a_x^4 + a_{x-1}^3 + 5a_{x-2}^2 + 3a_{x-3} = 0$$

solⁿ degree is 4

$$2) y_{h+3}^3 + 2y_{h+2}^2 + 2y_{h+1} = 0$$

solⁿ degree is 3

$$3) U_{x+4} + U_{x+1} + U_x = 0$$

80/2

Degree is 1

1

F

$$4) F(x+2h) + 4F(x+h) + 2F(h) = 0$$

80/1

Degree is 1

Linear Recurrence Relation:

A recurrence relation is called linear if it is of degree one.

ex:

$$a_{x+2} + 2a_x = 0$$

$$a_{x+2} + 2a_x = 0$$

In this equation the degree is one
then this equation is linear recurrence relation.

Note.

Non-linear Recurrence Relation:

A recurrence relation is called non-linear recurrence relation if it is of order degree - 2, 3, 4 - - -

Solving linear Recurrence relation?

1) If $a_n = a_{n-1} + 2a_{n-2}$ here $a_0 = 2$ and $a_1 = 7$ compute a_2 and a_3

80/1

$$a_n = a_{n-1} + 2a_{n-2}$$

$$a_0 = 2 \text{ and } a_1 = 7 \text{ then}$$

$$a_n = a_{n-1} + 2a_{n-2}$$

$$a_n = a_{n-1} + 2a_{n-2}$$

$$a_2 = a_{2-1} + 2a_{2-2}$$

$$a_2 = a_1 + 2a_0$$

$$a_2 = 7 + 2 \times 2$$

$$a_2 = 7 + 4$$

$$a_2 = 11$$

$$a_3 = a_{3-1} + 2a_{3-2}$$

$$a_3 = a_2 + 2a_1$$

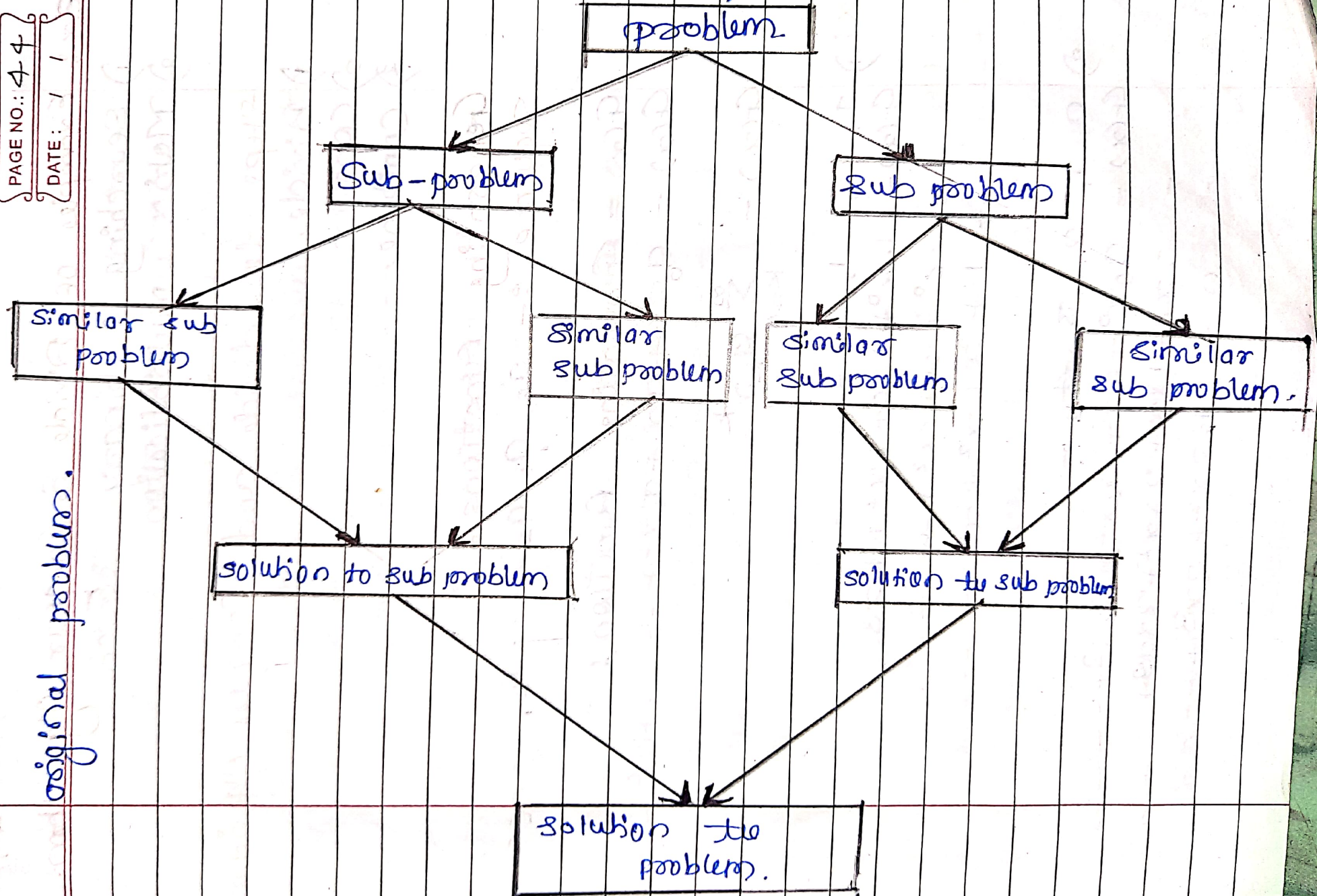
$$a_3 = 11 + 2 \times 7$$

$$a_3 = 11 + 14$$

$$a_3 = 25$$

Divide and Conquer Algorithms:-
Definition: Divide and Conquer is an algorithm design which by recursively breaking down a problem into sub problem of similar type until these become simple enough to be solved directly. The solution to the sub problems are then combined to give a solution to the

Divide & Conquer in action.



original problem.

Ex: Examples of Divide & Conquer Algorithm.

- 1) Searching : binary Search
- 2) Matrix multiplication

Steps to divide and Conquer Algorithm:

- 1) Divide.
- 2) Conquer
- 3) Combine.

Generating Functions:

Sequence :- $a_0, a_1, a_2, \dots, a_k, \dots$

$G(x)$ = Generating Function.

$$G(x) = a_0x^0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots$$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

Ex: 1) 1, 1, 1, - - - - -

$$G(x) = 1 \cdot x^0 + 1 \cdot x + 1 \cdot x^2 + 1 \cdot x^3 + \dots$$

$$= x^0 + x + x^2 + x^3 + \dots$$

$$= 1 + x + x^2 + x^3 + \dots$$

2) 0, 1, 2, 3, 4, - - - - -

$$G(x) = a_0x^0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$= 0x^0 + 1x + 2x^2 + 3x^3 + \dots$$

$$= 0 + x + 2x^2 + 3x^3 + \dots$$

3. $1, -a, a^2, -a^3$

Solⁿ $G(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots$

$G(x) = 1x^0 - ax + a^2x^2 - a^3x^3 + \dots$

$G(x) = 1 - ax + a^2x^2 - a^3x^3 + \dots$

closed form of generating function :

1) $1, 1, 1, 1, \dots$ OR $a_k = 1$

$G(x) = 1 + x + x^2 + x^3 + \dots$

Then

$G(x) = \frac{1}{1-x} \quad |x| < 1$

2) $2, 2, 2, \dots$ OR $a_k = 2$

$G(x) = 2 + 2x + 2x^2 + 2x^3 + \dots$

$G(x) = 2(1 + x + x^2 + x^3 + \dots)$

Then

$G(x) = \frac{2}{1-x}$

3) $1, 2, 3, 4, 5, \dots$ OR $a_k = k+1$

$G(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots \rightarrow (1)$

multiply both side by x

$xG(x) = x + 2x^2 + 3x^3 + 4x^4 + \dots \rightarrow (2)$

Subtract from (1) - (2)

$G(x) - xG(x) = 1 + (2x - x) + (3x^2 - 2x^2) + (4x^3 - 3x^3) + \dots$

$$G(x)(1-x) = 1+x+x^2+x^3+x^4+\dots$$

$$G(x)(1-x) = \frac{1}{1-x}$$

$$G(x) = \frac{1}{(1-x)^2}$$

4) $1, -1, 1, -1, \dots$ OR $a_k = (-1)^k$

$$G(x) = 1x^0 + (-1)x^1 + (1)x^2 + (-1)x^3 + \dots$$

$$G(x) = 1 - x + x^2 - x^3 + \dots$$

$$= 1 + (-x) + (-x^2)^2 + (-x^3)^3 + \dots$$

$$G(x) = \frac{1}{1+x}$$

5) $0, 1, 2, 3, \dots$ OR $a_k = k$

$$G(x) = 0 + x + 2x^2 + 3x^3 + \dots$$

$$G(x) = x[1 + 2x + 3x^2 + \dots]$$

$$G(x) = \frac{x}{(1-x)^2}$$

6) $0, 0, 1, 1, 1, \dots$

$$G(x) = 0x^0 + 0x^1 + 1x^2 + 1x^3 + \dots$$

$$G(x) = 0 + 0 + x^2 + x^3 + \dots$$

$$G(x) = x^2(1 + x + x^2 + \dots)$$

$$G(x) = \frac{x^2}{1-x}$$

7) 1, 1, 0, 1, 1, 1, ...

$$G(x) = 1x^0 + 1x^1 + 0x^2 + 1x^3 + 1x^4 + \dots$$

$$G(x) = (1 + x + x^2 + x^3 + \dots) - x^2 \rightarrow \textcircled{1}$$

in above eqn $\textcircled{1}$ add x^2 and sub $-x^2$

$$G(x) = [1 + x + \overset{\text{add}}{x^2} + x^3 + \dots] - \overset{\text{sub}}{x^2}$$

$$G(x) = \frac{1}{1-x} - x^2$$

8) 1, 0, -1, 0, 1, 0, -1, ...

$$G(x) = 1x^0 + 0x^1 - x^2 + 0x^3 + x^4 + \dots$$

$$G(x) = 1 + 0 - x^2 + 0 - x^4 + \dots$$

$$G(x) = 1 - x^2 + x^4 - x^6 + \dots$$

$$G(x) = \frac{1}{1+(x^2)}$$

$$G(x) = \frac{1}{1+x^2}$$

9) 3, -3, 3, -3, 3, ...

$$G(x) = 3 - 3x + 3x^2 - 3x^3 + 3x^4 + \dots$$

$$G(x) = 3(1 - x + x^2 - x^3 + x^4 - \dots)$$

$$G(x) = \frac{3}{1+x}$$

$$10) C(8,0), C(8,1), C(8,2), \dots, C(8,8), 0, 0, \dots$$

$$G(x) = C(8,0) + C(8,1)x + C(8,2)x^2 + \dots + C(8,8)x^8 + 0x^9 + \dots$$

$$G(x) = (1+x)^8$$

$$11) a_k = 2 + 3k.$$

$$G(x) = \frac{2}{1-x} + \frac{3x}{(1-x)^2} \quad k, 0, 1, 2, \dots$$

$$G(x) = \frac{2}{1-x} + \frac{3x}{(1-x)^2}$$

$$12) a_k = 4^k \quad \underline{\text{OR}} \quad 1 + 4x + 4^2x^2 + \dots$$

$$G(x) = \frac{1}{1-4x}$$

$$\text{Where } \frac{0}{1-\delta} \quad \delta = 4x$$

$$13) a_k = (-2)^k \quad \underline{\text{OR}} \quad 1 - 2x + (-2)^2x^2 + (-2)^3x^3 + \dots$$

$$G(x) = \frac{1}{1+2x}$$

S. NO.

 a^k $G(x)$

1

1

$$\frac{1}{1-x}$$

2

 $(-1)^k$

$$\frac{1}{1+x}$$

3

 c

$$\frac{c}{1-x}$$

4

 $k+1$

$$\frac{1}{(1-x)^2}$$

5

 k

$$\frac{x}{(1-x)^2}$$

6

 a^k

$$\frac{1}{1-ax}$$

7

 $(-a)^k$

$$\frac{1}{1+ax}$$

8

 k^2

$$\frac{x(1+x)}{(1-x)^3}$$

9.

 ka^k

$$\frac{ax}{(1-ax)^2}$$

Solution of Recurrence Relation by Using Generating Function:

Steps:

- 1) In given equation multiply by x^r (CF variable is x)
- 2) Take summation from $r=1$ to n if one initial condition is given
OR
 Take summation from $r=2$ to n if two initial condition is given.
- 3) Write each summation in terms of $G(x)$ or closed form.
- 4) Put values of each summation and find $G(x)$
- 5) Find partial fractions of $G(x)$.
- 6) Write as

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

Examples:

1) Solve $a_x - 2a_{x-1} - 3a_{x-2} = 0$ $a_0 = 3$
 $a_1 = 1$ by using generating function.

Soln: a_x

$a_x - 2a_{x-1} - 3a_{x-2} = 0$ and $a_0 = 3$
 and $a_1 = 1$

$$G(x) = a_x - 2a_{x-1} - 3a_{x-2} = 0 \rightarrow \textcircled{1}$$

Multiply by both side x^x

$$a_x x^x - 2a_{x-1} x^x - 3a_{x-2} x^x = 0$$

Multiply by both side Σ

$$\sum_{x=2}^{\infty} a_x x^x - 2 \sum_{x=2}^{\infty} a_{x-1} x^x - 3 \sum_{x=2}^{\infty} a_{x-2} x^x = 0 \rightarrow \textcircled{A}$$

$$\sum_{x=2}^{\infty} a_x x^x = a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \rightarrow \textcircled{B}$$

From Generating function

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

Then

$$G(x) - a_0 - a_1 x = a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

put these value in equation $\textcircled{A}$

$$\sum_{x=2}^{\infty} a_x x^x = G(x) - a_0 - a_1 x \rightarrow \textcircled{C}$$

$$\sum_{x=2}^{\infty} a_{x-1} x^x = a_{2-1} x^2 + a_{3-1} x^3 + a_{4-1} x^4 + \dots$$

$$\sum_{x=2}^{\infty} a_{x-2} x^x = a_{1-2} x^2 + a_{2-2} x^3 + a_{3-2} x^4 + \dots$$

Take common x .

$$\sum_{r=2}^{\infty} a_{r-1} x^r = x [a_1 x + a_2 x^2 + a_3 x^3 + \dots]$$

from Generating function

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$G(x) - a_0 = a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\boxed{\sum_{r=2}^{\infty} a_{r-1} x^r = x(G(x) - a_0)} \Rightarrow \textcircled{\text{II}}$$

$$\sum_{r=2}^{\infty} a_{r-2} x^r = a_{2-2} x^2 + a_{3-2} x^3 + a_{4-2} x^4 + \dots$$

$$\sum_{r=2}^{\infty} a_{r-2} x^r = a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots$$

Take common x^2

$$\sum_{r=2}^{\infty} a_{r-2} x^r = x^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$

from Generating function

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\boxed{\sum_{r=2}^{\infty} a_{r-2} x^r = x^2 G(x)} \Rightarrow \textcircled{\text{III}}$$

put the values of $\textcircled{\text{I}}$, $\textcircled{\text{II}}$ and $\textcircled{\text{III}}$ in equation $\textcircled{\text{A}}$

$$\begin{aligned} G(x) - a_0 - a_1 x - 2x [G(x) - a_0] - 3x^2 G(x) &= 0 \\ G(x) - a_0 - a_1 x - 2x G(x) + 2x a_0 - 3x^2 G(x) &= 0 \\ G(x) - 2x G(x) - 3x^2 G(x) &= a_0 + a_1 x - 2x a_0 \\ G(x) [1 - 2x - 3x^2] &= a_0 + a_1 x - 2a_0 x \end{aligned}$$

put the values of a_0 and a_1 in above equation

$$(1 - 2x - 3x^2)G(x) = 3 + 1 \cdot x - 2 \times 3x^2$$

$$(1 - 2x - 3x^2)G(x) = 3 + x - 6x^2$$

$$(1 - 2x - 3x^2)G(x) = 3 - 5x$$

$$G(x) = \frac{3 - 5x}{1 - 2x - 3x^2}$$

$$= \frac{-(3x^2 + 2x - 1)}{-(3x^2 + 3x - x - 1)}$$

$$= \frac{-[3x^2 + 3x - x - 1]}{-[3x(x+1) - 1(x+1)]}$$

$$= \frac{-[(x+1)(3x-1)]}{-[(x+1)(3x-1)]}$$

$$= 1$$

Then above equation

$$G(x) = \frac{3 - 5x}{-(x+1)(3x-1)} = \frac{-(5x-3)}{-(x+1)(3x-1)}$$

$$G(x) = \frac{5x-3}{(x+1)(3x-1)} \rightarrow \textcircled{B}$$

$$x+1=0$$

$$x = -1 \rightarrow \frac{5(-1)-3}{[3(-1)-1]}$$

$$= \frac{-5-3}{-3-1} = \frac{-8}{-4} = 2$$

$$3x-1=0$$

$$3x = 1 \rightarrow \frac{5(\frac{1}{3})-3}{(\frac{1}{3}+1)} = \frac{\frac{5}{3}-3}{\frac{1}{3}+1} = \frac{\frac{5-9}{3}}{\frac{4}{3}}$$

$$x = \frac{1}{3} \rightarrow \frac{\frac{5-9}{3}}{\frac{4}{3}} = \frac{-4}{4} = -1$$

$$= \frac{-4}{4} = -1$$

from equation B

Partial fraction

$$G(x) = \frac{2}{x+1} - \frac{1}{3x-1}$$

(omitted)

$$\frac{2}{1+x} + \frac{1}{1-3x}$$

$$a_x = 2(-1)^x + 3^x \rightarrow \frac{1}{1-ax} = \sum_{x=0}^{\infty} a^x$$

$$\frac{1}{1+x} = \sum_{x=0}^{\infty} (-1)^x$$

2) Solve $a_x - a_{x-1} - 2a_{x-2} = 0$ $a_0 = 0$
 $a_1 = 1$ by using generating function

soln

$$a_x - a_{x-1} - 2a_{x-2} = 0$$

$$a_x x^x - a_{x-1} x^x - 2a_{x-2} x^x = 0$$

$$\sum_{x=2}^{\infty} a_x x^x - \sum_{x=2}^{\infty} a_{x-1} x^x - 2 \sum_{x=2}^{\infty} a_{x-2} x^x = 0$$

$$\sum_{x=2}^{\infty} a_x x^x = a_2 x^2 + a_3 x^3 + \dots \rightarrow \textcircled{A}$$

$$\sum_{x=2}^{\infty} a_x x^x = a_2 x^2 + a_3 x^3 + \dots$$

$$\sum_{x=2}^{\infty} a_x x^x = G(x) - a_0 - a_1 x$$

put the values of a_0 and a_1

$$\sum_{x=2}^{\infty} a_x x^x = G(x) - 0 - 1x$$

$$= G(x) - x$$

$$\sum_{x=2}^{\infty} a_{x-1} x^x = a_1 x^2 + a_2 x^3 + \dots$$

$$\sum_{x=2}^{\infty} a_{x-1} x^x = x [a_1 x + a_2 x^2 + \dots]$$

$$\sum_{r=2}^{\infty} a_{r-1} x^r = x [G(x) - a_0]$$

$$= x [G(x) - 0]$$

$$\sum_{r=2}^{\infty} a_{r-1} x^r = x G(x)$$

$$\sum_{r=2}^{\infty} a_{r-2} x^r = a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots$$

$$= x^2 [a_0 + a_1 x + a_2 x^2 + \dots]$$

$$= x^2 G(x)$$

$$\sum_{r=2}^{\infty} a_{r-2} x^r = x^2 G(x)$$

putting this values in equation (A)

$$G(x) - x - x G(x) - 2(x^2 G(x))$$

$$G(x) - x - x G(x) - 2x^2 G(x)$$

$$G(x) - x G(x) - 2x^2 G(x) = x$$

$$G(x) (1 - x - 2x^2) = x$$

$$G(x) (1 - x - 2x^2) = \frac{x}{1 - x - 2x^2}$$

$$G(x) = \frac{x}{1 - x - 2x^2} \rightarrow \textcircled{1}$$

$$-(2x^2 + x - 1)$$

$$= [2x^2 + 2x - x - 1]$$

$$= [2x(x+1) - 1(x+1)]$$

$$= [(2x-1)(x+1)]$$

$$G(x) = \frac{x}{-(2x-1)(x+1)} = \frac{x}{(1-2x)(1+x)} \rightarrow \textcircled{2}$$

$$\begin{aligned} 1-2x &= 0 \\ -2x &= -1 \\ -x &= \frac{1}{2} \end{aligned}$$

$$= \frac{-1}{3(1-2x)} + \frac{-1}{3(1+x)}$$

$$G(x) = \frac{1}{3(1-2x)} - \frac{1}{3(1+x)}$$

$$1-2x=0$$

Form (2)

$$-2x = -1$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$

$$x = \frac{1}{2}$$

$$1 + \frac{1}{2}$$

$$\frac{2+1}{2}$$

$$= \frac{1}{2} \times \frac{2}{3}$$

$$= \frac{1}{3}$$

$$1+x=0$$

$$x = -1$$

$$\frac{-1}{1-2(-1)}$$

$$= \frac{-1}{1+2}$$

$$= \frac{-1}{3}$$

$$G(x) = \frac{1}{3(1-2x)} - \frac{1}{3(1+x)}$$

$$a_x = \frac{1}{3} a^x - \frac{1}{3} (-1)^x$$

$$a_x = \frac{1}{3} a^x + \frac{(-1)^x}{3}$$

Inclusion - Exclusion :-

* What is Inclusion - Exclusion?

Principle of Inclusion and Exclusion is an approach which derives the method of finding the number of elements in the union of two finite sets.

Consider two finite sets A and B. We can denote the principle of inclusion and exclusion formula as follows

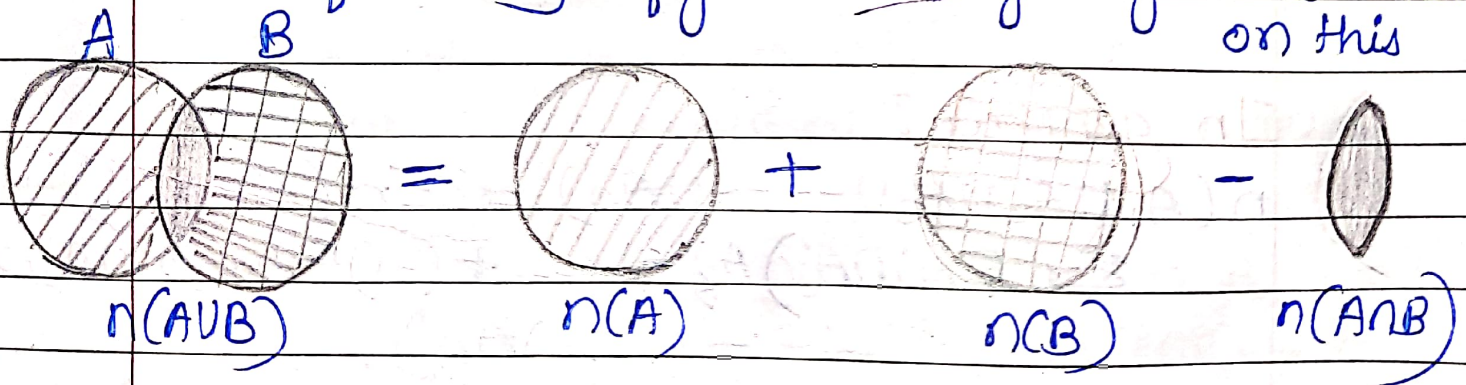
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Here $n(A)$ denotes the Cardinality of Set A.

$n(B)$ denotes the Cardinality of Set B. And $n(A \cap B)$ denotes the Cardinality of $(A \cap B)$

We have included A and B and excluded their common elements.

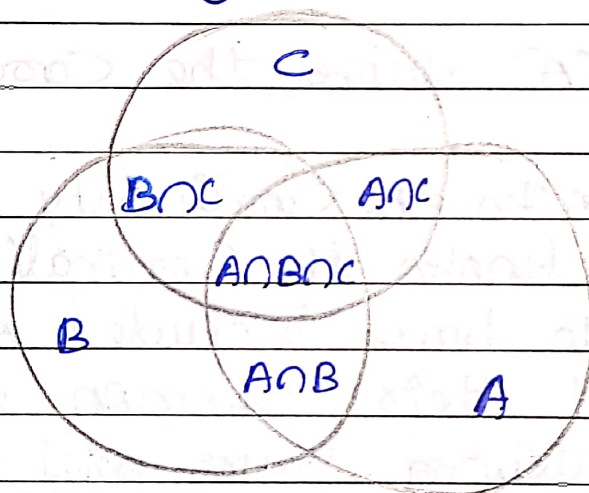
The following figure will give you more idea on this



If we have 3 sets A, B and C then according to the principle of inclusion and exclusion

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

The Principle of Inclusion and Exclusion represented by a Venn diagram for 3 sets is given below.



In general :

$$n(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n) = \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \dots + (-1)^{n-1} n(A_1 \cap A_2 \cap A_3 \dots \cap A_n).$$

Cardinality of a Set:

The Cardinality of a set is the number of element in a set.

Ex: Let $A = \{1, 2, 3, 6\}$.

The Cardinality in a set $A = 4$

Solve the Examples on the Principle of Inclusion and Exclusion:

Ex) Among a group of students 49 study Physics, 37 study English and 21 study Biology. If 9 of these students study Physics and English, 5 study English and Biology, 4 study Physics and Biology and 3 study Physics, English and Biology. Find the number of students in the group.

Solⁿ. Let P represent the number of student who study Physics, E represents the number of student who study English, and B represent the number of student who study Biology.

Number of student in the group
 $= n(P \cup E \cup B)$

Given $n(P) = 49$, $n(E) = 37$, $n(B) = 21$
 $n(P \cap E) = 9$, $n(E \cap B) = 5$, $n(P \cap B) = 4$
 $n(P \cap E \cap B) = 3$

$$n(P \cup E \cup B) = n(P) + n(E) + n(B) - n(P \cap E) - n(E \cap B) - n(P \cap B) + n(P \cap E \cap B)$$

$$= 49 + 37 + 21 - 9 - 5 - 4 + 3$$

$$= 92$$

Application of Inclusion - Exclusion:

- 1) This is used to solve combination and probability problems.
- 2) When it is necessary to find a counting method, which makes sure that an object is not counted twice.
- 3) The Construction of the chromatic polynomial of a graph.
- 4) Number of onto function
- 5) Permutation with for hidden positions
- 6) Bipartite graph perfect matching.
- 7) Stirling number of the second kind
- 8) Graph coloring.
- 9) Counting intersections.
- 10) Counting Derangements.